J. Fluid Mech. (1976), vol. 78, part 1, pp. 177–206
Printed in Great Britain

The stress field of slender particles oriented by a non-Newtonian extensional flow

By J. D. GODDARD

Department of Chemical Engineering, University of Southern California, Los Angeles

(Received 21 October 1975)

An analysis is presented of the creeping motion around a flow-oriented slender particle in a material medium subject to a uniaxial extension in the far field. A general quasi-steady rheological model is adopted, of a kind representing isotropic (Noll) fluids subject to time-independent velocity gradients, or isotropic solids subject to time-independent strain fields. The analysis is based on the premise of a shear-dominated motion in the near field, which is joined asymptotically to the extension-dominated motion in the far field. For axisymmetric particles, and to the order of terms in slenderness considered here, the far-field perturbation due to the particle can be represented as a distributed coaxial line force in a transversely isotropic medium whose strength is governed by the structure of the near-field rheology.

On the basis of the results for a single particle, a formula is derived for the stress contribution due to the presence of oriented slender fibres in dilute suspension in a non-Newtonian fluid. For certain simple rheological models exhibiting a strong shear-thinning behaviour, the particle contribution to tensile stress is greatly diminished relative to the Newtonian case, as was predicted by an earlier rudimentary treatment (Goddard 1975). The present analysis is thought to be highly promising for applications to general composite materials.

1. Introduction

The theoretical study of heterogeneous media has proved to be an interesting pursuit in the overall development of continuum physics and mechanics, and one whose influence extends beyond the applications to rheology contemplated here. In the present work we are concerned with a specific example of such a medium consisting of slender rigid particles suspended in a fluid.

It is already well known that slender oriented inclusions in a matrix exhibiting the linear rheological behaviour of the Hookean solid or the Newtonian fluid can have an enormous effect on the gross mechanical properties of the medium, which in the case of solids is relevant to the theory of fibre-reinforced composites (Russel & Acrivos 1972). In the case of Newtonian fluids, Batchelor (1971) has offered a theoretical analysis, well confirmed by subsequent experiments (Mewis & Metzner 1974; Weinberger & Goddard 1974), which predicts tensile stresses greatly in excess of those for the pure suspending fluid in suspensions of slender particles oriented by an extensional flow or 'pure-straining' motion. Besides the obvious analogy to the Hookean solid, this has implications for the rheology of such suspensions and more general fluids (Weinberger & Goddard 1974).

With these considerations in mind, it is interesting to inquire as to the possible influence which *nonlinear* rheological behaviour in the matrix can have on the rheological properties of the suspension, a subject which has already been taken up in a previous study (Goddard 1975, hereafter referred to as I). Apart from its direct relevance to suspensions of fibres in non-Newtonian fluids, such as 'fibre-loaded' polymer melts, the general problem arising from nonlinear mechanical response of the matrix material holds some practical interest for solid composites as well. In such systems it is known that various nonlinear effects, such as plastic yielding of the matrix in the vicinity of inclusions, can cause large departures from theories based on an assumed Hookean behaviour (Kardos 1973). Thus, in addressing ourselves to a problem involving a rheologically complex suspending fluid, we might reasonably hope to shed light on related problems involving complex solids.

The subject of suspensions of neutrally buoyant particles in incompressible liquids has received frequent attention in the recent literature, and within those confines we shall further restrict our discussion to the limit of infinitely dilute suspensions, where interaction between particles is considered to be entirely negligible. To deduce the properties of such suspensions it suffices, then, to treat the problem of a single particle or inclusion immersed in an infinite body of fluid.

Hence, with the usual assumption of particles small relative to the typical macroscopic length scales, we consider the case of a velocity field having constant velocity gradient at infinity, say

$$\left.\begin{aligned} &\mathbf{v} \to \mathbf{v}^{(0)} = \mathbf{\Gamma}^{(0)}.\mathbf{x}, \\ \text{with} \quad &\mathbf{\Gamma} \equiv (\nabla \mathbf{v})^{\mathrm{T}} \to \mathbf{\Gamma}^{(0)} = (\nabla \mathbf{v}^{(0)})^{\mathrm{T}}, \quad \text{a constant,} \end{aligned}\right\} \tag{1.1}$$

for $|\mathbf{x}| \to \infty$, where the superscript T denotes, here and below, the transpose of a second-rank tensor. Furthermore, we adopt the postulate of 'creeping' or inertialess flow, based on the assumed smallness of an appropriate Reynolds number for the particles, which leads to the governing equations

$$\nabla.\mathbf{T} = 0, \quad \nabla.\mathbf{v} = 0, \tag{1.2a, b}$$

together with (1.1) and the appropriate boundary conditions on the rigid particle surface. In addition, one must have a rheological equation relating the stress tensor $\mathbf{T}$ to the kinematics of the flow field, i.e. to $\mathbf{\Gamma}$.

We are interested here in the special case of elongated particles maintained in a steady orientation by a steady-state uniaxial extension of the form

$$\left.\begin{aligned} &\mathbf{v}^{(0)} = e_0[z\mathbf{i}_z - \tfrac{1}{2}r\mathbf{i}_r], \\ \text{with} \quad &\mathbf{\Gamma}^{(0)} = e_0[\mathbf{i}_z\mathbf{i}_z - \tfrac{1}{2}(\mathbf{i}_x\mathbf{i}_x + \mathbf{i}_y\mathbf{i}_y)] = e_0[\mathbf{i}_z\mathbf{i}_z - \tfrac{1}{2}(\mathbf{i}_r\mathbf{i}_r + \mathbf{i}_\theta\mathbf{i}_\theta)], \end{aligned}\right\} \tag{1.3}$$

where $e_0\,(> 0)$ denotes the extension rate and (x, y, z) and (r, θ, z) denote respectively Cartesian co-ordinates and the associated cylindrical-polar system, as indicated for the upper half-space in figure 1. We assume for simplicity that the particle is symmetrical with respect to the midplane $z = 0$ and hence that it

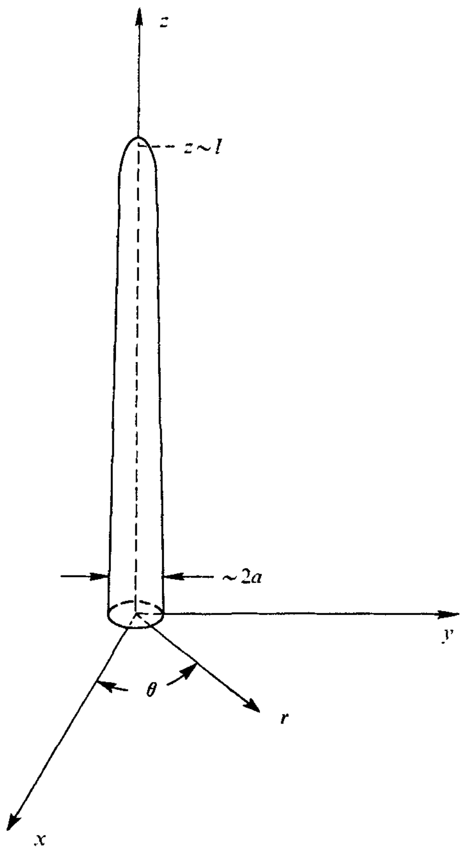

FIGURE 1. Upper-half space, showing upper half of particle and co-ordinates.

does not translate with respect to the fluid at infinity. Then, on the rigid stationary surface $\mathscr{P}$, say, of the particle we may assume the fluid velocity to vanish identically.

We are specifically interested here in the further simplifications that arise in the 'slender-body' limit, where, in terms of a particle aspect ratio α, we have that

$$\alpha \equiv l/a \to \infty, \tag{1.4}$$

a and l being characteristic particle dimensions (or semi-axes) in the radial (r) and axial (z) directions respectively. Also, the particle surface is assumed to possess a degree of smoothness, to be specified further below.

We recall that Batchelor (1971) has employed a refined version of the classical slender-body theory of Burgers (1938) to treat the Newtonian-fluid problem, and the present author (I) has recently discussed the corresponding non-Newtonian problem, which indicates the possibility of significant departures from the Newtonian-fluid results. While the analysis of the latter work was intended to provide an approximate treatment of closely spaced particles in non-dilute suspensions, it served also to emphasize the general need for a proper treatment of rheological complexity and far-field effects in related problems.

The present work considers such effects for an isolated particle in order both to improve the rudimentary slender-body theory proposed earlier for non-dilute suspensions and to provide a treatment of dilute suspensions.

As in the previous work we shall at the outset admit a rather general type of rheological behaviour for the fluid, characteristic of a Noll ('simple') fluid subject to a flow of constant stretch history (Truesdell 1966, p. 65; Pipkin 1972, p. 124). We recall that flows of this type are, by definition, such that their velocity gradient can be rendered time-independent when viewed from the appropriate frame of reference, and that they include many commonly studied flows, such as steady simple shear and the steady simple extension that constitutes the basic far-field flow in the present work.

For a broad class of fluid-like materials the stress in such flows reduces to a generally nonlinear tensor function of the velocity gradient. The adoption of this representation of stress in terms of an instantaneous velocity gradient for other flows amounts to an assumption of 'quasi-steady' rheological behaviour. As discussed in the appendix, the validity of this assumption, viewed as an approximation to the behaviour of a real fluid, depends on the nonlinear relaxation behaviour of the fluid and the material time rates of change associated with the particular flow. For a given fluid, this assumption is in principle amenable to further scrutiny, on the basis of an adequate rheological model. It will be adopted here without further apology beyond the observation that the solid-mechanics counterpart of our rheological representation enjoys a potentially much broader applicability.

In the next section we present a general statement of the essential features of the slender-body analysis to be employed here, which is based on consideration of an asymptotic shear-dominated flow in the near field and an extension-dominated flow in the far field. Because of the general nonlinear behaviour of the fluid, these asymptotic forms may be likened to a 'two-fluid' model of the system, with a hypothetical fluid in the near field differing from that in the far field. As with problems involving suspensions of one distinct fluid phase in another, care must be taken to ensure that stresses as well as kinematic variables match properly. However, in the present type of problem there is of course no distinct interface, and the near field must join smoothly onto the far field by means of a continuous rheological representation (as, for example, with miscible fluids).

In I it was assumed that a quasi-steady, shear-dominated flow in the near field would imply the validity of the viscometric-flow representation for the fluid rheology. However, a more careful consideration in the present work will show this to be true only immediately at the particle surface and, otherwise, only for rather special, 'shear-dominated' rheological models.

By means of the explicit rheological representations discussed in the appendix, we shall consider in some detail the case of axisymmetric particles. For this case, we shall construct a first-order outer or far-field solution as the flow perturbation to an anisotropic medium induced by a line of force singularities situated on the body axis, exactly as has been done for the analogous Newtonian-fluid and Hookean-solid problems (Cox 1970, 1971; Tillet 1970; Batchelor 1970*b*; Russel &

Acrivos 1972). Finally, after the appropriate matching with the relevant inner or near-field flow, we shall consider, in §4, the applications to dilute suspensions.

2. Slender-body analysis

2.1. *General form of the near-field approximation*

In line with the preceding remarks, and as discussed in the appendix, we assume a rheological equation of the form

$$\mathbf{T}(\mathbf{x}) = \mathbf{S}(\mathbf{\Gamma}(\mathbf{x})) - p(\mathbf{x})\,\mathbf{I} \tag{2.1}$$

for the stress field $\mathbf{T}(\mathbf{x})$ in terms of the velocity-gradient field $\mathbf{\Gamma}(\mathbf{x})$, as defined in (1.1). $\mathbf{S}$ denotes here a symmetric tensor function, defined on arbitrary (and generally asymmetric) tensor arguments $\mathbf{\Gamma}$, and p denotes a rheologically indeterminate pressure. At this point there is no real need to make explicit restrictions on the function $\mathbf{S}$ to account for fluid isotropy and 'frame-indifference', that is, for the proper dependence on superimposed rigid-body rotations. Accordingly, we defer such considerations to the appendix.

We are concerned now with certain asymptotic solutions $\mathbf{v}(\mathbf{x})$ to the field equations (1.2) augmented by the relations (2.1) and (1.1), the conditions (1.3) at infinity and, finally, the boundary condition

$$\mathbf{v}(\mathbf{x}) = 0, \quad \mathbf{x} \quad \text{on} \quad \mathscr{P}. \tag{2.2}$$

We henceforth assume these equations to be expressed in terms of dimensionless variables according to the scheme

$$(\mathbf{x}^*, \mathbf{v}^*, \mathbf{\Gamma}^*, \mathbf{T}^*) = (l\mathbf{x}, e_0 l\mathbf{v}, e_0\mathbf{\Gamma}, \mu^* e_0 \mathbf{T}), \tag{2.3}$$

where asterisks denote dimensional variables, $2l$ denotes the characteristic body length, e_0 the far-field extension rate of (1.1) and μ^* denotes a shear viscosity of the fluid, the shear viscosity corresponding to an assumed limiting Newtonian behaviour for $\mathbf{\Gamma} \to 0$.

On the basis of the usual slender-body analysis for $\alpha \to \infty$ and the discussion in I, we expect to encounter a set of non-uniformly valid asymptotic approximations for the velocity field, of the 'singular-perturbation' variety. With that expectation, we consider separately a set of near-field or 'inner' approximations and a set of far-field or 'outer' approximations for the velocity and stress fields and the governing equations.

In the near field, we use the decomposition

$$\mathbf{x} = \mathbf{r} + z\mathbf{i}_z, \quad \text{with} \quad \mathbf{r} \equiv x\mathbf{i}_x + y\mathbf{i}_y \equiv r\mathbf{i}_r, \tag{2.4}$$

and adopt the stretched co-ordinates

$$\bar{\mathbf{r}} = \alpha\mathbf{r}, \quad \text{i.e.} \quad (\bar{x}, \bar{y}, \bar{r}) = (\alpha x, \alpha y, \alpha r), \tag{2.5}$$

which all remain $O(1)$ near the body surface for $\alpha \to \infty$. The gradient operator ∇ then assumes the form

$$\nabla \equiv \nabla_0 + \nabla_1 = \alpha\overline{\nabla}_0 + \nabla_1, \tag{2.6}$$

where
$$\overline{\nabla}_0 \equiv \frac{1}{\alpha}\nabla_0 \equiv \mathbf{i}_x\frac{\partial}{\partial\bar{x}} + \mathbf{i}_y\frac{\partial}{\partial\bar{y}} \equiv \mathbf{i}_r\frac{\partial}{\partial\bar{r}} + \frac{\mathbf{i}_\theta}{\bar{r}}\frac{\partial}{\partial\theta}$$

and
$$\nabla_1 \equiv \mathbf{i}_z\partial/\partial z,$$

and, in terms of these variables, the velocity gradient in (1.1) becomes

$$\mathbf{\Gamma} = \bar{\mathbf{\Gamma}}_0 + \bar{\mathbf{\Gamma}}_1, \tag{2.7}$$

where
$$\bar{\mathbf{\Gamma}}_0 = \alpha(\bar{\nabla}_0 \mathbf{v})^{\mathrm{T}}, \quad \bar{\mathbf{\Gamma}}_1 = (\nabla_1 \mathbf{v})^{\mathrm{T}}.$$

Thus the field equations (1.2) become

$$\bar{\nabla}_0 . \mathbf{T} = -\alpha^{-1}\nabla_1 . \mathbf{T}, \quad \bar{\nabla}_0 . \mathbf{v} = -\alpha^{-1}\nabla_1 . \mathbf{v}. \tag{2.8}$$

The structure of (2.7) and (2.8) suggests that, as asymptotic forms for $\alpha \to \infty$ with, say, $\mathbf{v} \to \bar{\mathbf{v}}^{(0)}(\alpha, \bar{\mathbf{r}}, z)$, one will have

$$\left.\begin{aligned} \mathbf{\Gamma} &\to \bar{\mathbf{\Gamma}}^{(0)} \sim \alpha(\bar{\nabla}_0 \bar{\mathbf{v}}^{(0)})^{\mathrm{T}} \\ \mathbf{T} &\to \bar{\mathbf{T}}^{(0)} \sim \mathbf{S}(\bar{\mathbf{\Gamma}}^{(0)}) - \bar{p}^{(0)}\mathbf{I}, \end{aligned}\right\} \tag{2.9}$$

and with

$$\bar{\nabla}_0 . \bar{\mathbf{T}}^{(0)} = 0, \quad \bar{\nabla}_0 . \bar{\mathbf{v}}^{(0)} = 0. \tag{2.10}$$

In addition, $\bar{\mathbf{v}}^{(0)}$ must satisfy the boundary condition (2.2) at the particle surface and match with an appropriate far-field velocity distribution for $\bar{\mathbf{r}} \to \infty$.

The precise form of the limiting velocity field $\bar{\mathbf{v}}^{(0)}$ will depend crucially on the nature of the rheological function $\mathbf{S}$ in (2.9). In I it was concluded that, in certain cases, there could exist solutions $\bar{\mathbf{v}}^{(0)} = O(1)$ which would match with the asymptotic form (1.3) of the unperturbed flow,

$$\mathbf{v}^{(0)} = z\mathbf{i}_z - (r/2\alpha)\,\mathbf{i}_r \equiv z\mathbf{i}_z - \bar{\mathbf{r}}/2\alpha \to z\mathbf{i}_z \tag{2.11}$$

for $\alpha \to \infty$, which is in distinct contrast to the Newtonian case (Batchelor 1971). However, the previous analysis was restricted to the zeroth-order term $\bar{\mathbf{v}}^{(0)}$, and no detailed description of the far field was given.

In order to provide a theory involving both the near field and the far field, we might attempt to develop a set of near-field asymptotic expansions or perturbation series for the velocity field and the associated kinematic and dynamic variables. Because of the generally nonlinear nature of the rheological equation (2.1), and the difficulty of establishing, *ab initio*, a connexion between the perturbation series for the various field quantities, we choose to formulate the analysis in terms of a set of 'successive approximations' for the quantities $\mathbf{v}, \mathbf{\Gamma}, \ldots$, say $\bar{\mathbf{v}}^{(k)}, \bar{\mathbf{\Gamma}}^{(k)}, \ldots$, for $k = 0, 1, 2, \ldots$. Although we shall soon concentrate on the axisymmetric problem, it is thought worthwhile to consider a general scheme that might be applicable to more complex flows and might also be more easily grasped. (The reader who finds otherwise can proceed directly to §2.3.)

Thus, at any given level $\bar{\mathbf{v}}^{(k)}$ of approximation for the velocity field we have a corresponding velocity gradient

$$\bar{\mathbf{\Gamma}}^{(k)} = \bar{\mathbf{\Gamma}}_0^{(k)} + \bar{\mathbf{\Gamma}}_1^{(k)}, \tag{2.12}$$

which is derived from $\bar{\mathbf{v}}^{(k)}$ as indicated by (2.7) and which is formally dominated by the term $\bar{\mathbf{\Gamma}}_0^{(k)}$. Then, in proceeding from one level of approximation $\mathbf{\Gamma}^{(k-1)}$ to the next, i.e. $\mathbf{\Gamma}^{(k)}$, we expect that the change in the approximation to $\mathbf{\Gamma}$, namely

$$\Delta\bar{\mathbf{\Gamma}}^{(k)} \equiv \bar{\mathbf{\Gamma}}^{(k)} - \bar{\mathbf{\Gamma}}^{(k-1)}, \tag{2.13}$$

will be dominated by the 'component'

$$\Delta\bar{\mathbf{\Gamma}}_0^{(k)} \equiv \bar{\mathbf{\Gamma}}_0^{(k)} - \bar{\mathbf{\Gamma}}_0^{(k-1)}. \tag{2.14}$$

Accordingly, we take for the stress function $\mathbf{S}$ in (2.1) an approximation of the form

$$\bar{\mathbf{S}}^{(k)} = \mathbf{S}(\bar{\mathbf{\Gamma}}^{(k)}) \doteq \bar{\mathbf{S}}^{(k-1)} + \bar{\boldsymbol{\eta}}^{(k-1)} : [\Delta\bar{\mathbf{\Gamma}}_0^{(k)}]^{\mathrm{T}}, \tag{2.15}$$

where $\bar{\boldsymbol{\eta}}$ denotes a fourth-rank tensor derivable from a general fourth-rank 'viscosity' tensor defined by

$$\boldsymbol{\eta} = \boldsymbol{\eta}(\mathbf{\Gamma}) = \partial\mathbf{S}/\partial\mathbf{\Gamma}^{\mathrm{T}} \quad (\text{or } \eta_{ijkl} = \partial S_{ij}/\partial\Gamma_{lk}). \tag{2.16}$$

In (2.15), $\boldsymbol{\eta}$ is of course to be evaluated at $\bar{\mathbf{\Gamma}}^{(k-1)}$. Also, the 'double-dot' product in (2.15) indicates the ordered contraction of the last two tensoral indices of $\boldsymbol{\eta}$ with the (two) indices of $[\Delta\mathbf{\Gamma}_0^{(k)}]^{\mathrm{T}}$.

Now, given approximations $\bar{\mathbf{v}}^{(k-1)}$ and $\bar{\mathbf{T}}^{(k-1)}$, we assume that the succeeding pair $\bar{\mathbf{v}}^{(k)}$ and $\bar{\mathbf{T}}^{(k)}$, for $k = 1, 2, \ldots$, are governed by

$$\bar{\nabla}_0 . \mathbf{T}^{(k)} = -\alpha^{-1}\nabla_1 . \bar{\mathbf{T}}^{(k-1)}, \quad \bar{\nabla}_0 . \bar{\mathbf{v}}^{(k)} = -\alpha^{-1}\nabla_1 . \bar{\mathbf{v}}^{(k-1)}, \tag{2.17}$$

which are derived from (2.8). As suggested by (2.15) and (2.1), we assume that $\Delta\mathbf{\Gamma}_0^{(k)}$ is related to the stress field by

$$\bar{\boldsymbol{\eta}}^{(k-1)} : [\Delta\bar{\mathbf{\Gamma}}_0^{(k)}]^{\mathrm{T}} = \Delta\bar{\mathbf{T}}^{(k)} + \Delta\bar{p}^{(k)}\mathbf{I}, \tag{2.18}$$

where

$$\Delta\bar{p}^{(k)} = \bar{p}^{(k)} - \bar{p}^{(k-1)}, \quad \Delta\bar{\mathbf{T}}^{(k)} = \bar{\mathbf{T}}^{(k)} - \bar{\mathbf{T}}^{(k-1)}.$$

On combining (2.17) and (2.18), one obtains, finally, a set of inhomogeneous linear equations for $\bar{\mathbf{v}}^{(k)}$ and $\bar{p}^{(k)}$

$$\bar{\nabla}_0 . [\bar{\boldsymbol{\eta}} : \bar{\nabla}_0 \bar{\mathbf{v}}'] = \bar{\nabla}_0 \bar{p}' + \bar{\mathbf{b}}, \quad \bar{\nabla}_0 . \bar{\mathbf{v}}' = \bar{q}, \tag{2.19}$$

with

$$\bar{\mathbf{v}}' \equiv \Delta\bar{\mathbf{v}}^{(k)}, \quad \bar{p}' \equiv \alpha^{-1}\Delta\bar{p}^{(k)}, \quad \bar{\boldsymbol{\eta}} \equiv \bar{\boldsymbol{\eta}}^{(k-1)},$$

and with the inhomogeneous terms determined from the lower-level approximations by

$$\left.\begin{aligned} \bar{\mathbf{b}} &\equiv \bar{\mathbf{b}}^{(k-1)} = -\alpha^{-1}[\bar{\nabla}_0 + \alpha^{-1}\nabla_1] . \bar{\mathbf{T}}^{(k-1)} \equiv -\alpha^{-2}\nabla . \bar{\mathbf{T}}^{(k-1)} \\ \text{and} \quad \bar{q} &\equiv \bar{q}^{(k-1)} = -\alpha^{-1}[\bar{\nabla}_0 + \alpha^{-1}\nabla_1] . \bar{\mathbf{v}}^{(k-1)} \equiv -\alpha^{-2}\nabla . \bar{\mathbf{v}}^{(k-1)} \end{aligned}\right\} \tag{2.20}$$

for $k = 1, 2, \ldots$. The governing equations for $k = 0$ are to be obtained from equations like (2.9) and (2.10). The exact form of these equations will depend crucially on the fluid rheology, a matter to be reconsidered in a simpler version of the problem to follow. It is also necessary to place certain rheological restrictions on the derivatives η_{ijkl} in relations like (2.15), to ensure that the stress perturbations arising from the dominant velocity gradient are in turn dominant, and we shall henceforth assume that such restrictions are satisfied by our fluid.

Then we require the near-field approximation at any level $k = 0, 1, 2, \ldots$, to satisfy the boundary conditions (2.2) at the particle surface and to match in an appropriately defined way with a proper far-field approximation. As we shall see below, some simplification of this otherwise complex type of problem is possible in the case of an axisymmetric flow, when the zeroth-order flow field $\bar{\mathbf{v}}^{(0)}$ is a rectilinear-shearing or 'viscometric' flow. In this case, both the operator ∇_0 and the tensor $\boldsymbol{\eta}$ can be replaced by simple scalar forms.

2.2. *The far-field approximation*

In the 'far-field', $r^2+z^2 = O(1)$ for $\alpha \to \infty$, we assume that the flow can be represented as a hierarchy of perturbations to the basic or 'unperturbed' extensional flow $\mathbf{v}^{(0)}$ in (1.3). Thus, by means of a relation analogous to (2.15), we take

$$\mathbf{S}^{(k)} = \mathbf{S}^{(k-1)} + \boldsymbol{\eta}^{(k-1)} : [\Delta\boldsymbol{\Gamma}^{(k)}]^{\mathrm{T}}, \tag{2.21}$$

which involves the tensor appearing in (2.15), now evaluated at $\boldsymbol{\Gamma}^{(k-1)}$. In this way, one obtains a set of field equations analogous to (2.20):

$$\nabla . [\boldsymbol{\eta} : \nabla \mathbf{v}'] = \nabla p' + \mathbf{b}, \quad \nabla . \mathbf{v}' = 0, \tag{2.22}$$

with, now,

$$\mathbf{v}' \equiv \Delta \mathbf{v}^{(k)}, \quad p' \equiv \Delta p^{(k)}, \quad \boldsymbol{\eta} \equiv \boldsymbol{\eta}^{(k-1)}, \quad \mathbf{b} \equiv \mathbf{b}^{(k-1)} = -\nabla . \mathbf{T}^{(k-1)}.$$

In the case of the first-order approximation ($k = 1$), we note that equation (2.22) for $\mathbf{v}^{(1)}$ has $\mathbf{b}^{(0)} \equiv 0$ and, hence, is homogeneous. Here we suppose the basic flow in the far field to be characterized by a uniform state of stress,

$$\mathbf{T}^{(0)} = \mathbf{S}(\boldsymbol{\Gamma}^{(0)}) - p^{(0)}\mathbf{I}, \quad p^{(0)} = \text{constant}, \tag{2.23}$$

as well as a *constant* tensor

$$\boldsymbol{\eta}^{(0)} \equiv \boldsymbol{\eta}(\boldsymbol{\Gamma}^{(0)}), \tag{2.24}$$

with

$$\boldsymbol{\Gamma}^{(0)} = \mathbf{i}_z\mathbf{i}_z - \tfrac{1}{2}(\mathbf{i}_r\mathbf{i}_r + \mathbf{i}_\theta\mathbf{i}_\theta)$$

representing the basic far-field gradient of (1.3).

Hence the field equations for $k = 1$ represent the motion of a homogeneous anisotropic medium and, as such, are amenable to certain solution methods from the theory of anisotropic elasticity. In particular, because of the axial symmetry of the unperturbed flow, the tensor $\boldsymbol{\eta}^{(0)}$ is endowed with the symmetry of a transversely isotropic medium, which will permit us presently to construct the first-order perturbation to the far-field flow due to the body.

2.3. *The axisymmetric case*

The near-field form of the velocity field $\mathbf{v}^{(0)}$ in (2.11) suggests that the asymptotic form of the near-field flow can provisionally be taken as purely axial with

$$\bar{\mathbf{v}}^{(0)} = \bar{u}^{(0)}(\alpha; \bar{\mathbf{r}}, z)\,\mathbf{i}_z + o(1) \tag{2.25}$$

for $\alpha \to \infty$, where $\bar{u}^{(0)}$ is $O(1)$. With (2.6), this implies an asymptotic velocity gradient of the form

$$\begin{aligned} \boldsymbol{\Gamma} \sim \bar{\boldsymbol{\Gamma}}^{(0)} &= \alpha[\mathbf{i}_z \bar{\nabla}_0 \bar{u}^{(0)} + o(1)] \\ &\equiv \alpha[\mathbf{i}_z\mathbf{i}_r\, \partial\bar{u}^{(0)}/\partial\bar{r} + \mathbf{i}_z\mathbf{i}_\theta\, \bar{r}^{-1}\partial\bar{u}^{(0)}/\partial\theta + o(1)]. \end{aligned} \tag{2.26}$$

To terms $o(1)$ relative to the leading term, this represents a rectilinear shearing flow and suggests that the near-field flow is 'quasi-viscometric', provided that the rheological function $\mathbf{S}$ in (2.1) is such that the terms $o(1)$ have negligible influence on the stress, for $\alpha \to \infty$. However, the latter condition does not ensure the validity of (2.25), for, as discussed in the earlier analysis (I), there still remains the well-known incompatibility of general rectilinear flows, other than planar or axisymmetric, with the equations of motion (2.10) (cf. Green & Rivlin 1956; Ericksen 1958; Dodson, Townsend & Walters 1974, who give other references). Therefore, to avoid the possibility of transverse motions of magnitude $O(1)$ for $\alpha \to \infty$, while retaining an element of generality in our rheological assumptions, we shall henceforth limit our discussion to the axisymmetric situation. In this case, (2.25) and (2.26) can be retained, with of course $\bar{\mathbf{r}}$ replaced by $\bar{r}$ in (2.25) and $\partial/\partial\theta$ taken as zero in (2.26) and elsewhere as appropriate.

At this point, it is worthwhile to reconsider briefly the complete flow problem, since its axisymmetric version can be written out in a more explicit form, without resorting to any of the approximations employed in our slender-body analysis. Thus, for the axisymmetric problem, the velocity field is given by

$$\mathbf{v} = u(r,z)\,\mathbf{i}_z + v(r,z)\,\mathbf{i}_r, \tag{2.27}$$

where u and v denote the axial and radial components respectively. Therefore the velocity gradient is

$$\mathbf{\Gamma} = e_z\mathbf{i}_z\mathbf{i}_z + \gamma_z\mathbf{i}_z\mathbf{i}_r + \gamma_r\mathbf{i}_r\mathbf{i}_z + e_r\mathbf{i}_r\mathbf{i}_r + e_\theta\mathbf{i}_\theta\mathbf{i}_\theta, \tag{2.28}$$

where, for brevity, we employ the notation

$$e_z \equiv \Gamma_{zz} = \partial u/\partial z, \quad \gamma_z \equiv \Gamma_{zr} = \partial u/\partial r,$$
$$\gamma_r \equiv \Gamma_{rz} = \partial v/\partial z, \quad e_r \equiv \Gamma_{rr} = \partial v/\partial r, \quad e_\theta \equiv \Gamma_{\theta\theta} = v/r$$

for the components of the velocity gradient. The e's are of course connected by the continuity equation

$$e_z + e_r + e_\theta \equiv \nabla\,.\,\mathbf{v} = \frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0. \tag{2.29}$$

As shown in the appendix, for a quasi-steady velocity gradient of the form (2.28), the rheological function $\mathbf{S}$ is given, up to an additive isotropic pressure, by

$$\mathbf{S} = (\mathbf{i}_r\mathbf{i}_z + \mathbf{i}_z\mathbf{i}_r)\,t + \mathbf{i}_z\mathbf{i}_z s_1 + \mathbf{i}_r\mathbf{i}_r s_2, \tag{2.30}$$

where t, s_1 and s_2 are functions of the velocity gradients in (2.28). Accordingly, the stresses are given as functions of position by

$$\left.\begin{aligned} \tau(r,z) &\equiv T_{rz} &&\equiv S_{rz} = t(\gamma_z,\gamma_r,e_z,e_r), \\ \sigma_z(r,z) &\equiv T_{zz} - T_{\theta\theta} &&\equiv S_{zz} = s_1(\dots\dots\dots) \\ \text{and}\qquad \sigma_r(r,z) &\equiv T_{rr} - T_{\theta\theta} &&\equiv S_{rr} = s_2(\dots\dots\dots). \end{aligned}\right\} \tag{2.31}$$

Certain salient properties of these rheological functions are discussed in the appendix, where, also, $\mathbf{S}(\mathbf{\Gamma})$ is expressed as an explicit tensor function of $\mathbf{\Gamma}$. In the following analysis it will be useful to keep in mind certain properties to be attributed to the fluid behaviour in the limit of the basic far-field flow, where $\gamma_z \to 0$, $\gamma_r \to 0$, $e_z \to 1$ and $e_r \to -\frac{1}{2}$, and for which

$$\left.\begin{aligned} t \to 0, \quad s_1 \to \text{constant} \geqslant 0, \quad s_2 \to 0 \\ \text{and}\qquad \partial t/\partial e \to 0, \quad \partial s/\partial\gamma \to 0. \end{aligned}\right\} \tag{2.32}$$

The last two relations, in which s, γ and e refer to any of the variables denoted by these letters in (2.28) and (2.31), require a degree of smoothness in the rheological behaviour, as mentioned in the appendix.

For the present purposes, the representations (2.30)–(2.32) suffice and the stress equations (1.2a) take on the well-known, simple form

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(r\tau)+\frac{\partial \sigma_z}{\partial z} &= \frac{\partial p}{\partial z},\\ \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)+\frac{\partial \tau}{\partial z} &= \frac{\partial p}{\partial r},\end{aligned}\right\} \tag{2.33}$$

which, together with (2.28)–(2.31), constitute a set of second-order nonlinear differential equations for u, v and p. The boundary condition at the (axisymmetric) particle surface now assumes the elementary form

$$u(r,z) = v(r,z) = 0 \quad \text{for} \quad r = \hat{r}(z)/\alpha, \tag{2.34}$$

where $\hat{r}(z)/\alpha$ denotes the radius of revolution of the surface, and $\hat{r}(z)$ and its z derivatives are assumed to be $O(1)$ for $|z| < 1$.

Returning to the slender-body analysis, we write (2.17) as

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}[\bar{r}\bar{\tau}^{(k)}] = \frac{1}{\alpha}\frac{\partial}{\partial z}[\bar{p}^{(k-1)}-\bar{\sigma}_z{}^{(k-1)}], \tag{2.35a}$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}[\bar{r}\bar{\sigma}_r^{(k)}]-\frac{\partial \bar{p}^{(k)}}{\partial \bar{r}} = -\frac{1}{\alpha}\frac{\partial \bar{\tau}^{(k-1)}}{\partial z}, \tag{2.35b}$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}[\bar{r}\bar{v}^{(k)}]+\frac{1}{\alpha}\frac{\partial \bar{u}^{(k)}}{\partial z} = 0, \tag{2.35c}$$

for $k = 0, 1, 2, \ldots$, where the right-hand sides of the first two equations are to be set equal to zero in the zeroth-order approximation, $k = 0$.

For (2.35c), we have chosen a form which is at variance with the general approximation scheme (2.17)–(2.19) and (2.20) in which there may exist non-trivial solutions to $\nabla_0 . \mathbf{v} = 0$ that do not involve sources or sinks of mass. With the present one-dimensional form for ∇_0, such is not the case, and our choice of the form of (2.35c) will provide for a well-defined zeroth approximation, thereby eliminating certain needless algebraic iterations. Also, it will facilitate the later discussion of matching with the far field, where terms like

$$v^{(0)} = -\tfrac{1}{2}r = -\bar{r}/2\alpha,$$

which are $O(1/\alpha)$ in the near field, become $O(1)$.

Now, (2.35a) can be integrated immediately and, for the case $k = 0$, yields the elementary expression (cf. I)

$$\bar{\tau}^{(0)}(\alpha;\bar{r},z) = \hat{\tau}^{(0)}(\alpha;z)\,\hat{r}(z)/\bar{r}, \tag{2.36}$$

where $\hat{\tau}^{(0)}$ is the (as yet unknown) shear stress at the particle surface $\bar{r} = \hat{r}(z)$. When combined with the continuity equation (2.35c) and the rheological equation (2.31) for t, (2.36) gives a set of implicit partial differential equations in u and v of the form

$$t\left(\alpha\frac{\partial u}{\partial \bar{r}}, \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, \alpha\frac{\partial v}{\partial \bar{r}}\right) = \bar{\tau}^{(0)}(\alpha; \bar{r}, z), \tag{2.37}$$

involving α as a parameter.

Obviously, the form of the function t will govern the asymptotic relation between u, v and τ for $\alpha \to \infty$, and we desire asymptotic solutions $\bar{u}^{(0)}$ and $\bar{v}^{(0)}$ to these equations whose asymptotic matching with the far field will serve to determine the form of $\bar{\tau}^{(0)}$ in (2.36). Thus, instead of (2.37), we require of the zeroth-order approximations $\bar{\tau}^{(0)}$, $\bar{u}^{(0)}$ and $\bar{v}^{(0)}$ that, for fixed but otherwise arbitrary values of $\bar{r}$ and z,

$$\lim_{\alpha \to \infty} [\bar{\tau}^{(0)}(\alpha; \bar{r}, z) - \bar{t}^{(0)}] = 0, \tag{2.38}$$

where

$$\bar{t}^{(0)} = t(\bar{\gamma}_z^{(0)}, \bar{\gamma}_r^{(0)}, \bar{e}_z^{(0)}, \bar{e}_r^{(0)}) \equiv t\left(\alpha\frac{\partial \bar{u}^{(0)}}{\partial \bar{r}}, \frac{\partial \bar{v}^{(0)}}{\partial z}, \frac{\partial \bar{u}^{(0)}}{\partial z}, \alpha\frac{\partial \bar{v}^{(0)}}{\partial \bar{r}}\right), \tag{2.39}$$

which we expect generally to yield a differential equation, involving $\bar{\tau}^{(0)}$, $\bar{u}^{(0)}$ and $\bar{v}^{(0)}$.

We further require that $\bar{u}^{(0)}$ and $\bar{v}^{(0)}$ satisfy the boundary condition (2.34) and, whenever the form of the rheological equation permits, that they match asymptotically for $\bar{r} \to \infty$ with the components of the basic far-field velocity $\mathbf{v}^{(0)}$. For arbitrary rheological models the latter condition is by no means ensured and, indeed, is *not* satisfied by a Newtonian fluid.

On the other hand, whenever the required zeroth-order velocities exist, then we shall have, by means of the further rheological relations (2.31), expressions of the form

$$\bar{\sigma}^{(0)} = \bar{s}^{(0)} \equiv s(\bar{\gamma}_z^{(0)}, \bar{\gamma}_r^{(0)}, \bar{e}_z^{(0)}, \bar{e}_r^{(0)}) \tag{2.40}$$

for the stresses $\bar{\sigma}_r^{(0)}$ and $\bar{\sigma}_z^{(0)}$ in (2.35). Thus, by (2.35) and (2.40), one obtains for the zeroth-order pressure field

$$\bar{p}^{(0)} = \bar{s}_2^{(0)} + \int_{\bar{r}}^{\bar{r}} \bar{s}_2^{(0)} \frac{d\bar{r}}{\bar{r}} + \text{a function of } z. \tag{2.41}$$

Having established, whenever feasible, a set of zeroth-order approximations for $\bar{\tau}^{(0)}$, $\bar{u}^{(0)}$, $\bar{v}^{(0)}$ and $\bar{p}^{(0)}$, one could in principle proceed to the calculation of higher-order approximations. Thus (2.40) and (2.35a) with $k = 1$ give an immediate integral for $\bar{\tau}^{(1)}$:

$$\bar{\tau}^{(1)} = \frac{1}{\alpha\bar{r}} \int_{\bar{r}}^{\bar{r}} \frac{\partial}{\partial z}[\bar{p}^{(0)} - \bar{s}_1^{(0)}]\,\bar{r}\,d\bar{r} + \bar{\tau}^{(0)}. \tag{2.42}$$

Here we have chosen the additive homogeneous solution for $\bar{\tau}^{(1)}$ to be the identical with $\bar{\tau}^{(0)}$ of (2.36). This condition, together with the matching of (2.42) to the far field in § 3, provides in effect the definition of $\bar{\tau}^{(0)}$.

As regards the first-order velocity and pressure fields, we write, in the spirit of the scheme outlined in § 2.1,

$$\bar{\mathbf{v}}^{(1)} = \bar{\mathbf{v}}^{(0)} + \Delta\bar{\mathbf{v}}^{(1)}, \tag{2.43}$$

obtaining then from the appropriate form of (2.15)

$$\partial(\Delta\bar{u}^{(1)})/\partial\bar{r} = (\bar{\tau}^{(1)} - \bar{t}^{(0)})\,(\alpha\,\partial\bar{t}^{(0)}/\partial\gamma_z)^{-1}, \tag{2.44}$$

where $\bar{t}^{(0)}$ and $\partial\bar{t}^{(0)}/\partial\gamma_z$ denote the values of t and its derivative at $\mathbf{v} = \bar{\mathbf{v}}^{(0)}$ and $\bar{\tau}^{(1)}$ is given by (2.42). The relevant equations for $\bar{p}^{(1)}$ and $\bar{v}^{(1)}$ are then to be obtained from (2.35b, c), with $k = 1$, together with (2.36).

However, our intention here is not to investigate fully the nature of such higher-order terms, so that the asymptotic forms of (2.41)–(2.44) will suffice for the matching of velocity and stress at $\bar{r} = \infty$ to be reconsidered in §3.

For certain types of rheology, in particular for a Newtonian fluid, a zeroth-order solution satisfying the matching requirement laid down above will not be possible. By way of contrast, we consider now a certain class of non-Newtonian fluids whose shear behaviour allows for solutions which are $O(1)$ in the near field for $\alpha \to \infty$ but which display a weak body influence in the far field.

2.4. *Shear-dominated behaviour*

In I, the near-field stress was taken to be viscometric in character for $\alpha \to \infty$, which implies that the shear-stress function t in (2.31) behaves like

$$t(\gamma_z, \gamma_r, e_z, e_r) \sim T(\gamma_z) \equiv t(\gamma_z, 0, 0, 0) \tag{2.45}$$

for $\gamma_z \to \infty$ with γ_r, e_z and e_r fixed, where T denotes the viscometric shear-stress function. As can be gathered from the discussion in the appendix, the behaviour (2.45) must be regarded as special for it implies a special type of functional dependence of t on the scalar invariants formed from the velocity gradients in (2.28), a dependence which might be termed 'shear-dominated' as indicated in the above heading. If such behaviour is assumed, and if it is further assumed that the fluid exhibits strong 'shear thinning', with (apparent viscosity)

$$T(\gamma)/\gamma \to 0 \quad \text{for} \quad \gamma \to \infty, \tag{2.46}$$

then, as also shown in the analysis just referred to, it is possible to find asymptotic solutions $\bar{\mathbf{v}}^{(0)}$ of the kind discussed in the preceding subsection.

To establish this more carefully, and to provide a result which may be easily comprehended, let us further assume (i) an invertible function $T(\gamma)$, with $\gamma = G(\tau)$ denoting the inverse for $\tau = T(\gamma)$, and (ii) the limiting Newtonian behaviour at small shear rates:

$$T(\gamma) \to \gamma \quad \text{or} \quad G(\tau) \to \tau \qquad \text{for} \qquad \gamma \quad \text{or} \quad \tau \to 0, \tag{2.47}$$

where a dimensional scaling like that of (2.3) is implied. Then, because of (2.46), (2.38) is satisfied by taking

$$\alpha\, \partial\bar{u}^{(0)}/\partial\bar{r} \equiv \bar{\gamma}_z^{(0)} = G(\bar{\tau}^{(0)}) - \bar{\tau}^{(0)}. \tag{2.48}$$

Hence, on integration and changing the variable from $\bar{r}$ to $\bar{\tau}^{(0)}$ according to (2.36), we have

$$\bar{u}^{(0)}(\alpha; \bar{r}, z) = \frac{\hat{\tau}^{(0)}\hat{r}}{\alpha} \int_{\bar{\tau}^{(0)}}^{\hat{\tau}^{(0)}} h(\tau)\, d\tau, \tag{2.49}$$

where

$$h(\tau) = (G(\tau) - \tau)/\tau^2$$

and $\bar{\tau}^{(0)}$ is given by (2.36). The device of subtracting the argument τ from $G(\tau)$ in (2.48) and (2.49) serves to avoid a logarithmic 'Newtonian' singularity, which represents the shear behaviour of the near-field rather than that required for matching of both velocity *and* stress in the far field.

In § 3, we shall see that it is precisely the first-order perturbation $\Delta\bar{u}^{(1)}$, governed by the asymptotic form of (2.44), which serves to accomplish the matching. Furthermore, the very form of (2.49) serves to emphasize that our 'zeroth-order' velocity in the near field owes its existence to departures from Newtonian behaviour. While the above device is doubtless not the only way of achieving the desired result, just as the functional form (2.48) for $\bar{\gamma}_z^{(0)}$ is not the only one possible, it nevertheless produces a zeroth-order approximation that appears to have the desired properties.

In particular, the requirement that (2.49) match for $\bar{r} \to \infty$ with the far-field velocity $u^{(0)} = z$ becomes

$$\frac{\alpha z}{\hat{r}(z)} = \tau^{(0)} \int_0^{\tau^{(0)}} h(\tau)\, d\tau, \tag{2.50}$$

which provides an implicit equation for $\tau^{(0)}(\alpha; z)$, a function which is determined by the form of $T(\gamma)$ and which, incidentally, is seen to depend on the single variable $\alpha z/\hat{r}(z)$.

For example, let us employ a minor but appropriate modification of a well-known empiricism, the so-called 'power-law' fluid, by taking

$$G(\tau) = (m-1)\beta\tau^m + \tau \quad (\tau > 0), \tag{2.51}$$

where β and m are constants with $\beta(m-1) \geqslant 0$ and $m > 1$ for shear-thinning behaviour, while $m = 1$ in the limit of Newtonian behaviour. Then (2.50) and (2.49) become

$$\left.\begin{aligned} \tau^{(0)}(\alpha; z) &= [\alpha|z|/\beta\hat{r}(z)]^{1/m}\,\mathrm{sgn}\, z \\ \text{and} \qquad \bar{u}^{(0)} &= z[1-(\hat{r}/\bar{r})^{m-1}] \end{aligned}\right\} \tag{2.52}$$

for $m > 1$. These equations are but slight variations of equations presented earlier in I. It can be seen that the postulate (2.45) and the condition at $\bar{r} = \infty$ are both satisfied for $m > 1$ but not for $m = 1$.

For a more general viscometric behaviour, the correspondingly more general form (2.49) applies, although, strictly speaking, we must consider it limited to the shear-dominated behaviour (2.45). This relation might nevertheless serve in several instances to provide a useful zeroth-order approximation $\bar{u}^{(0)}$ in the near field since the flow must be exactly of the simple-shearing type at the particle surface, where $e_r = e_z = \gamma_r \equiv 0$, as with any incompressible flow near a no-slip boundary, and whatever the value of the parameter α.

We consider next the first-order perturbation in the far field and its matching with the near field. This will provide, in the form of equation (3.47), a more complete asymptotic formula than (2.50) and will highlight the non-uniform nature of representations like (2.51) and (2.52).

3. First-order perturbation in the far field and matching with the near field

3.1. *Solution by potential functions*

As pointed out in § 2.2, the governing equations (2.21) for the first-order approximation in the far field have a form representative of a general anisotropic medium with a constant material tensor $\boldsymbol{\eta}^{(0)}$ relating stress to velocity gradient. As such, they are similar in form to those governing small strains superimposed on large deformations in elastic bodies, as discussed extensively in the treatise of Green & Zerna 1954, hereafter referred to as G & Z). This important fact permits us to adopt certain methods from solid mechanics to construct a solution to the problem at hand.

First of all, we note that, in the present context, the tensor $\boldsymbol{\eta}^{(0)}$ must possess the symmetry of the basic uniaxial extension $\boldsymbol{\Gamma}^{(0)}$ in (1.3) and (2.23), which is that associated with 'transverse isotropy' (Love 1944, pp. 151 ff., G & Z, pp. 130 ff.). As a consequence, the stresses are linearly related to the velocity gradients according to the following array of constant coefficients, which of course determines the form of $\boldsymbol{\eta}^{(0)}$:

$$\begin{array}{c|ccc|cccccc} & E_{zz} & E_{rr} & E_{\theta\theta} & E_{zr} & E_{z\theta} & E_{r\theta} & \Omega_{rz} & \Omega_{\theta z} & \Omega_{r\theta} \\ T_{zz} & 4\mu'-2\mu'' & 0 & 0 & & & & & & \\ T_{rr} & 0 & 2\mu' & 2\mu'-2\mu'' & & 0 & & & 0 & \\ T_{\theta\theta} & 0 & 2\mu'-2\mu'' & 2\mu' & & & & & & \\ T_{zr} & & & & 2\mu & 0 & 0 & 2\nu & 0 & 0 \\ T_{z\theta} & & 0 & & 0 & 2\mu & 0 & 0 & 2\nu & 0 \\ T_{r\theta} & & & & 0 & 0 & 2\mu'' & 0 & 0 & 0 \end{array} \tag{3.1}$$

Here the E's and Ω's denote, respectively, the components (in the z, r, θ system in § 2) of the symmetric rate-of-deformation tensor $\mathbf{E}$ and the antisymmetric vorticity tensor $\boldsymbol{\Omega}$:

$$\mathbf{E} = \mathbf{E}^{\mathrm{T}} \equiv \tfrac{1}{2}(\boldsymbol{\Gamma}+\boldsymbol{\Gamma}^{\mathrm{T}}), \quad \boldsymbol{\Omega} = -\boldsymbol{\Omega}^{\mathrm{T}} = \tfrac{1}{2}(\boldsymbol{\Gamma}-\boldsymbol{\Gamma}^{\mathrm{T}}), \tag{3.2}$$

in which we recall that $\boldsymbol{\Gamma}$ is to be interpreted, in the notation of § 2.2, as the perturbation $\Delta\boldsymbol{\Gamma}^{(1)}$ to the basic far-field gradient $\boldsymbol{\Gamma}^{(0)}$. Accordingly, we recall that the explicit dependence of the stress in (3.1) on vorticity refers to the tensor $\Delta\boldsymbol{\Omega}^{(1)}$, thereby implying a proper material response to superposed rotation.

In (3.1) we have adopted a deviatoric or traceless form of the relation given by G & Z (pp. 130 ff.), as is appropriate for an incompressible medium, where the stresses are defined only up to an arbitrary isotropic pressure. In the appendix, the relation of the coefficients in (3.1) to the axisymmetric rheological functions t, s_1 and s_2 in (2.30) is briefly discussed.

We shall be mainly concerned with the case of axisymmetric flows. This symmetry admits a class of solutions derivable from a 'potential' (G & Z, pp. 130 ff., pp. 182 ff.), say $\phi(k, r, z)$, as

$$\left.\begin{aligned} \mathbf{v} &= (\nabla_0 + k\nabla_1)\,\phi, \\ \text{that is,}\quad u = k\frac{\partial\phi}{\partial z}, \quad v &= \frac{\partial\phi}{\partial r} \quad \left(\text{with}\quad w = v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \equiv 0 \quad \text{here}\right), \end{aligned}\right\} \tag{3.3}$$

in which case, to satisfy (2.22), one must require that the pressure p and the potential function ϕ satisfy

$$p = k[4\mu' - 2\mu'' - (\mu - \nu)k - (\mu + \nu)]\,\partial^2\phi/\partial z^2 + \text{constant} \tag{3.4}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + k\frac{\partial^2\phi}{\partial z^2} = 0, \tag{3.5}$$

where k is equal to one of the two roots, say $k_\pm$, of the characteristic equation

$$(\mu - \nu)k^2 + 2(\mu'' + \mu - 3\mu')k + (\mu + \nu) = 0. \tag{3.6}$$

We shall not bother to record here the corresponding formulae, in terms of ϕ, for the velocity gradients and the stresses, which follow from (3.3), (3.1) and (3.4) and which can also be deduced from formulae given by G & Z (pp. 133 and 182).

For a linear isotropic solid material or a very special simple fluid (a 'Reiner–Rivlin' or 'Stokesian fluid'; see the appendix) one may set $\nu = 0$, in which case

$$k_+ = b + (b^2 - 1)^{\frac{1}{2}}, \quad k_- = k_+^{-1} = b - (b^2 - 1)^{\frac{1}{2}},$$

where $b = (3\mu' - \mu - \mu'')/\mu$. Then, in the fully degenerate case of an isotropic (Hookean) solid or (Newtonian) fluid, with $\mu'' = \mu' = \mu$, these reduce further to

$$k_+ = k_- = b = 1. \tag{3.7}$$

In the non-degenerate case, a general solution of the form (3.3) can (as a slight variation on G & Z) be expressed as

$$\phi = \phi_+ - \phi_-, \quad v = \partial(\phi_+ - \phi_-)/\partial r \tag{3.8a, b}$$

and

$$u = \frac{\partial}{\partial z}(k_+\phi_+ - k_-\phi_-) = k_+^{\frac{1}{2}}\left(\frac{\partial\phi_+}{\partial z_+}\right) - k_-^{\frac{1}{2}}\left(\frac{\partial\phi_-}{\partial z_-}\right) \tag{3.8c}$$

where, with subscripts $\pm$ taken separately,

$$z_\pm = z/k_\pm^{\frac{1}{2}} \tag{3.9}$$

and the functions $\phi_\pm(r, z)$ represent general harmonic functions of r and $z_\pm$ satisfying the axisymmetric Laplace equation

$$\nabla_\pm^2\phi_\pm \equiv \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z_\pm^2}\right]\phi_\pm = 0. \tag{3.10}$$

The pressure field is accordingly to be obtained from (3.4) by means of linear combinations like those in (3.8).

Now, as an observation that is perhaps unique to the present work, we note that, by what is essentially 'D'Alembert's method' for ordinary differential equations (see, for example, Ince 1956, pp. 136 ff.), one can construct certain solutions to (2.22) which remain valid in the isotropic limit (3.7). Thus, to general solutions of the form (3.8), we merely add a solution in which $\phi_\pm$ are chosen to be the *same* harmonic function; that is, we let

$$\phi_\pm = \Phi(r, z_\pm)/\Delta, \tag{3.11}$$

where $\Delta = \Delta(k_+, k_-)$, independent of r and z, is a function of $k_\pm$ such that $\Delta \to 0$ for $k_+ \to k_- \to 1$ and the limit

$$\chi(r,z) \equiv \lim\,[\Phi(r,z_+) - \Phi(r,z_-)]/\Delta \tag{3.12}$$

exists. Then, in view of the fact that

$$\nabla_+^2 \nabla_-^2 (\phi_+ - \phi_-) \equiv \nabla_-^2 \nabla_+^2 (\phi_+ - \phi_-) = 0 \tag{3.13}$$

for distinct roots $k_\pm$, one has formally in the limit $k_+ \to k_- \to 1$, when $\nabla_+^2 \to \nabla_-^2 \to \nabla^2$, that the limit function χ is biharmonic:

$$\nabla^4 \chi = 0. \tag{3.14}$$

To be definite here, we take

$$\Delta \equiv k_+ - k_-, \tag{3.15}$$

so that, by (3.12),

$$\chi = -\tfrac{1}{2} z\, \partial\Phi(r,z)/\partial z. \tag{3.16}$$

The associated velocity fields are, in the same limit, given by (3.8) as

$$v = \frac{\partial\chi}{\partial r} = -\tfrac{1}{2} z \frac{\partial^2\Phi}{\partial r\,\partial z}, \quad u = \tfrac{1}{2}\left(\frac{\partial\Phi}{\partial z} - z\frac{\partial^2\Phi}{\partial z^2}\right), \tag{3.17}$$

from which we recover a well-known classical representation of solutions for the isotropic problem (cf. Lamb 1945, p. 604; or Happel & Brenner 1965, p. 221). We note in passing that the above method of deriving solutions for isotropic media, as the limit of 'potential' solutions for anisotropic media, is by no means restricted to the axisymmetric motions and incompressible media considered here.

For the present analysis, the axisymmetric forms (3.8) and the corresponding isotropic limits in (3.17) will suffice. Furthermore, we are mainly concerned here with special potential solutions, of a kind that will satisfy (3.2) and match asymptotically for $r \to 0$ with the outer limit of the near-field solutions.

As in the corresponding Newtonian or Hookean problem (Cox 1970; Batchelor 1970*b*; Russell & Acrivos 1972), it is our hypothesis here that the far-field perturbations $\Delta p^{(1)}$ and $\Delta \mathbf{v}^{(1)}$ can be represented by a line-force singularity along the body axis $r = 0$, $|z| < 1$. Hence, by superposition, we can construct the appropriate potential function and related fields from a point-force singularity.

3.2. *The coaxial point-force and line-force singularities*

The construction of fundamental singular solutions for linear anisotropic media appears to be an exceedingly difficult problem of classical continuum mechanics, even though it is solved in principle by the Volterra–Synge contour-integral representation (Synge 1957, p. 411). In the simplest non-trivial case, which is the transversely isotropic medium of interest here, Willis (1965) has reduced this representation to purely algebraic terms, but the resulting expressions are still of a rather formidable complexity.† Hence, for a coaxial force collinear with the

† Thus the present author was unable, with a reasonable expenditure of effort, to reduce the expressions of Willis to the forms given here for the axisymmetric case.

axis of material symmetry, we shall employ the method of potential solutions discussed in the preceding subsection to construct a simpler expression. In particular, letting

$$R = R(r,z) \equiv |\mathbf{x}| = (r^2+z^2)^{\frac{1}{2}} \tag{3.18}$$

henceforth denote the distance from the origin $r = z = 0$, we assert that the solution which corresponds to a unit point force acting at the origin in the $+z$ direction is obtained from (3.8), (3.11) and (3.15) by taking†

$$\Phi(r,z) = \frac{1}{8\pi(\mu-\nu)} \ln\left(\frac{R+z}{R-z}\right), \tag{3.19}$$

and that the velocity fields, derived from it by means of (3.8), are then

$$u = U(r,z) \equiv \frac{1}{4\pi(k_+-k_-)(\mu-\nu)}\left(\frac{k_+^{\frac{1}{2}}}{R_+} - \frac{k_-^{\frac{1}{2}}}{R_-}\right) \tag{3.20}$$

and

$$v = V(r,z) \equiv -\frac{1}{4\pi(k_+-k_-)(\mu-\nu)\,r}\left(\frac{z_+}{R_+} - \frac{z_-}{R_-}\right) \tag{3.21}$$

where $R_\pm(r,z) = R(r,z_\pm)$ with $z_\pm$ defined as in (3.9). By means of the stress formulae to be presented below as (3.25)–(3.27), this assertion can be verified in the standard way, by derivation of the traction on an arbitrary closed surface enclosing the origin (which is most conveniently chosen here to be a circular cylinder with axis on $r = 0$). Also, it can be verified that (3.20) and (3.21) have the proper isotropic limits, which are also given directly by (3.17) and (3.19).

By means of the fundamental solutions (3.19)–(3.21) and the superposition principle one can, of course, obtain the velocity and stress fields associated with an axial distribution of force $dF(z)$ on $r = 0$, which, when differentiable as will be assumed here, can be written as

$$dF(z) = f(z)\,dz, \quad \text{with} \quad f(z) = dF(z)/dz. \tag{3.22}$$

Thus, for a line distribution confined to $|z| < 1$, the associated potential function is obtained in terms of (3.19) from (3.8) and

$$\phi_\pm = \int_{-1}^{+1} \Phi(r, z_\pm - z_\pm^*) f(z^*)\,dz^*/(k_+-k_-). \tag{3.23}$$

The velocity fields are then given by (3.8) or, alternatively, in terms of (3.20) and (3.21), by formulae like

$$u(r,z) = \int_{-1}^{+1} U(r, z-z^*) f(z^*)\,dz^*. \tag{3.24}$$

We record here the formulae for the pressure (3.4) and the stress components (3.1) corresponding to a coaxial point force at the origin:

$$p = P(r,z), \quad T_{ij} = \Sigma_{ij}(r,z) - P(r,z)\,\delta_{ij}, \tag{3.25}$$

† It can be seen that (3.19) is a superposition of the potential functions $\ln(R\pm z)$ of isotropic elasticity associated with a so-called 'line of dilatation' (Love 1944, p. 188) along $r = 0, z \lessgtr 0$.

with

$$\left.\begin{aligned} P &= -A^{-1}[B_+J_+ - B_-J_-], \\ \Sigma_{\theta\theta} \equiv -(\Sigma_{zz}+\Sigma_{rr}), \quad \Sigma_{zz} &= -(4\mu'-2\mu'')A^{-1}[J_+-J_-], \\ \Sigma_{rr} &= A^{-1}\{2\mu''r^{-2}[H_+-H_-]+2\mu'[J_+-J_-]\}, \\ \Sigma_{zr} \equiv \Sigma_{rz} &= -(\mu-\nu)rA^{-1}[k_+^{\frac{1}{2}}K_+ - k_-^{\frac{1}{2}}K_-], \end{aligned}\right\} \tag{3.26}$$

where

$$\left.\begin{aligned} A = 4\pi(\mu-\nu)(k_+-k_-), \quad B_\pm &= (\mu+\nu)+[(\mu-\nu)-2\mu']k_\pm, \\ H_\pm \equiv z_\pm/R_\pm, \quad J_\pm &\equiv z_\pm/R_\pm^3, \quad K_\pm \equiv 1/R_\pm^3 \end{aligned}\right\} \tag{3.27}$$

and $R_\pm$ are defined as in (3.20)–(3.21).

Superposition formulae of the type (3.24) will thus involve integrals having the general form

$$\int_{-1}^{+1} \frac{(z_\pm - z_\pm^*)^n f(z^*)}{R^m(r, z_\pm - z_\pm^*)}\,dz^* = k_\pm^{\frac{1}{2}(m-n)} \int_{-1}^{+1} \frac{(z-z^*)^n f(z^*)\,dz^*}{R^m(r_\pm, z-z^*)}, \tag{3.28}$$

with $r_\pm = k_\pm^{\frac{1}{2}} r$ and R defined by (3.9). The singular behaviour of this type of integral for $r \to 0$ has been studied extensively in connexion with various slender-body theories (e.g. by Fraenkel 1969; Cox 1970, 1971; Tillet 1970; Russel & Acrivos 1972, to name but a few).† For the terms of main interest here, we have that

$$\int_{-1}^{+1} \frac{f(z^*)}{R(r, z-z^*)}\,dz^* \sim -2f(z)\ln r, \tag{3.29}$$

$$\int_{-1}^{+1} \frac{f(z^*)}{R^3(r, z-z^*)}\,dz^* \sim (2/r^2) f(z) \tag{3.30}$$

and

$$\int_{-1}^{+1} \frac{(z-z^*) f(z^*)}{R^3(r, z-z^*)}\,dz^* \sim 2f'(z)\ln r \tag{3.31}$$

for $r \to 0$ with $0 < r^2 \ll 1 - z^2$ (see Russel & Acrivos 1972). In the present context, the force distribution, denoted by $f(z)$ in (3.22) and (3.29)–(3.31), is also a function of α, say $f(\alpha; z)$, whose form is to be ascertained from the following considerations.

3.3. *Matching with the near field*

Here we wish to show that the type of first-order approximation constructed above in §3.2 for the far field can be joined asymptotically with those derived from the first-order approximations for the near field developed in §2.3.

Considering first the shear-stress terms τ, we obtain, by means of the function Σ_{rz} in (3.26) and the asymptotic form (3.30), the far-field relation

$$\tau^{(1)} \equiv T_{rz}^{(1)} \sim -f(\alpha; z)/2\pi r \quad \text{for} \quad r \to 0. \tag{3.32}$$

On the other hand, under the assumption that the integral in (2.41) is of a smaller order than the term $\bar{\tau}^{(0)}$ for $\alpha \to \infty$, we have for the near-field stress that

$$\bar{\tau}^{(1)} \sim \bar{\tau}^{(0)} \equiv \hat{\tau}^{(0)}(\alpha; z)\,\hat{r}(z)/\bar{r} \tag{3.33}$$

† It is interesting to observe that the type of potential solutions arising here for the anisotropic medium serve to establish a certain kinship to other, diverse physical applications of slender-body analyses, a kinship which is somewhat obscured by the specialization to isotropic media.

for $\alpha \to \infty$ at fixed $\bar{r}$. Then, through the matching of the fundamental singularity in (3.32) and (3.33), the axial force distribution is given by

$$f(\alpha; z) = -2\pi\alpha^{-1}\hat{t}^{(0)}(\alpha; z)\,\hat{r}(z), \tag{3.34}$$

which is the elementary force balance on the particle surface proposed in I.

Next we show that (3.34) is sufficient to ensure that the corresponding axial velocities also match. The appropriate far-field form is obtained from (3.20), (3.24) and (3.29) as

$$\Delta u^{(1)} \equiv u^{(1)} - u^{(0)} \sim \frac{-f(\alpha; z)}{2\pi(\mu-\nu)}\ln r. \tag{3.35}$$

The near-field perturbation $\Delta\bar{u}^{(1)}$ is to be obtained from (2.43). Then, making use of (3.33) and (3.34) together with rheological relations of the form

$$t^{(0)} \to 0, \quad \partial t^{(0)}/\partial\gamma_z \to \mu - \nu \quad \text{for} \quad \bar{r} \to \infty,$$

obtained from (2.32) and equations (A 9)–(A 12) of the appendix, one can readily verify that (3.35) indeed matches with $\Delta\bar{u}^{(1)}$.

Incidentally, we note that, even if there exists no zeroth-order near-field flow having the requisite properties, the equation (2.44) for $\bar{u}^{(1)} \equiv \Delta\bar{u}^{(1)}$ can still remain valid, upon setting $\bar{u}^{(0)} \equiv 0$ and $\bar{t}^{(0)} \equiv 0$ in both (2.42) and (2.44). Thus, for example, we recall that one finds for the Newtonian case that

$$\bar{u}^{(1)} = \frac{\hat{t}^{(0)}\hat{r}}{\alpha}\ln\frac{\bar{r}}{\hat{r}} \sim z - \frac{f}{2\pi}\ln r$$

for $\bar{r} \to \infty$, and hence that

$$\frac{f}{2\pi} = -\frac{z}{\ln\alpha}\left\{1 + O\left(\frac{1}{\ln\alpha}\right)\right\}.$$

Therefore, in this case, the basic far-field velocity $u^{(0)} = z$ is matched to the *first-order* near-field velocity.

In any event, we have confirmed that the matching of shear stress implies the matching of axial velocity and conversely. At the same time, we have established in (3.34) the order-of-magnitude relation

$$f(\alpha; z) = O\{\hat{t}^{(0)}(\alpha; z)/\alpha\} \quad \text{for} \quad \alpha \to \infty, \tag{3.36}$$

which also serves as a criterion for the validity of our far-field perturbation analysis. In particular, both in the preceding analysis and in that to follow, we must require that $f = O(1/\ln\alpha)$ for $\alpha \to \infty$.

While the remaining stresses are not as easily reckoned with, we can nevertheless establish a result similar in essence to that obtained above, namely that the matching of radial velocity components implies the matching of the radial stress components.

First, we note that to the order of terms considered here the asymptotic form of the far-field perturbation $\Delta v^{(1)}$ for $r \to 0$ can readily be obtained from the continuity equation and the relation (3.35) for $\Delta u^{(1)}$ as

$$\Delta v^{(1)} \sim [2\pi(\mu-\nu)]^{-1} f' r \ln r + O(r), \tag{3.37}$$

where f' denotes the z derivative of f. This agrees asymptotically with the form of $\bar{v}^{(1)}$ deduced from (2.35) and (2.44).

Next, we note that, with the stress convention of § 2 ($p \equiv -T_{\theta\theta}$), (2.35*b*) can be rearranged to read

$$\bar{T}^{(1)}_{rr} = \bar{s}^{(1)}_2 - \bar{p}^{(1)} = -\int_{\hat{r}}^{\bar{r}} \bar{s}^{(1)}_2 \frac{d\bar{r}}{\bar{r}} - \frac{1}{\alpha}\frac{d}{dz}(\hat{\tau}^{(0)}\hat{r}) \ln \bar{r} + \text{a function of } z. \quad (3.38)$$

To show that this matches appropriately with the far-field stress $T^{(1)}_{rr}$, we recall that the rheological term in the integrand is defined by

$$\bar{s}^{(1)}_2 = s_2\left(\alpha\frac{\partial \bar{u}^{(1)}}{\partial \bar{r}}, \frac{\partial \bar{v}^{(1)}}{\partial z}, \frac{\partial \bar{u}^{(1)}}{\partial z}, \alpha\frac{\partial \bar{v}^{(1)}}{\partial \bar{r}}\right). \quad (3.39)$$

Accordingly, we employ the corresponding far-field expression

$$s^{(1)}_2 = s_2\left(\frac{\partial u^{(1)}}{\partial r}, \frac{\partial v^{(1)}}{\partial z}, \frac{\partial u^{(1)}}{\partial z}, \frac{\partial v^{(1)}}{\partial r}\right) \quad (3.40)$$

to cast (3.38) into the form

$$\bar{T}^{(1)}_{rr} = \int_{\hat{r}}^{\bar{r}} [s^{(1)}_2 - \bar{s}^{(1)}_2] \frac{d\bar{r}}{\bar{r}} - \int_{\hat{r}/\alpha}^{r} s^{(1)}_2 \frac{dr}{r} - \frac{1}{\alpha}\frac{d}{dz}(\tau^{(0)}\hat{r}) \ln \bar{r} + \text{a function of } z. \quad (3.41)$$

In this way, we expect the first integral in (3.41) to converge for $\bar{r} \to \infty$ provided that $\bar{\mathbf{v}}^{(1)} \to \mathbf{v}^{(1)}$. It remains then to show, with account taken of the other terms in (3.41), that the expression given will match properly with the far-field stress.

Now we note that the arguments of s_2 in (3.40) are given asymptotically for small r by

$$\left.\begin{aligned} \gamma^{(1)}_z \equiv \frac{\partial u^{(1)}}{\partial r} \sim c\frac{f}{r}, \quad \gamma^{(1)}_r \equiv \frac{\partial v^{(1)}}{\partial z} \sim cf'' r \ln r, \\ e^{(1)}_z \equiv \frac{\partial u^{(1)}}{\partial z} \sim 1 - cf' \ln r, \quad e^{(1)}_r \equiv \frac{\partial v^{(1)}}{\partial r} \sim -\tfrac{1}{2} + cf' \ln r, \end{aligned}\right\} \quad (3.42)$$

where $c \equiv 1/2\pi(\mu - \nu)$. If we assume that f and its z derivatives are all $O(1/\ln \alpha)$ as $\alpha \to \infty$, then we may consider these as $o(1)$ perturbations to the far-field gradients. For $r = O(1)$, this is already assumed tacitly in the perturbation analysis of § 2.2, and it is furthermore necessary for the continuation into the near field $r \equiv \bar{r}/\alpha = O(1/\alpha)$ assumed here for the purposes of matching.

With that stated, we take $s^{(1)}_2$, as in (3.1), to be

$$s^{(1)}_2 = 2\mu''(\Delta E^{(1)}_{rr} - \Delta E^{(1)}_{\theta\theta}) \equiv 2\mu''\left(\frac{\partial v^{(1)}}{\partial r} - \frac{v^{(1)}}{r}\right) = 2\mu'' r \frac{\partial}{\partial r}\left(\frac{\Delta v^{(1)}}{r}\right), \quad (3.43)$$

which, on substitution into the second integral of (3.41), gives

$$\bar{T}^{(1)}_{rr} = \int_{\hat{r}}^{\bar{r}} [s^{(1)}_2 - \bar{s}^{(1)}_2] \frac{d\bar{r}}{\bar{r}} - 2\mu'' \frac{\Delta v^{(1)}}{r} + \frac{1}{2\pi} f' \ln \bar{r} + \text{a function of } z. \quad (3.44)$$

It remains, finally, to compare the asymptotic form of the far-field stress $T^{(1)}_{rr}$ with (3.44). By means of the relations (3.25)–(3.27), one sees that the superposition integral for $T^{(1)}_{rr}$ is determined by

$$\Sigma_{rr} - P = A^{-1}\{2\mu'' r^{-2}[H_+ - H_-] + [C_+ J_+ - C_- J_-]\}, \quad (3.45)$$

where $C_\pm = 2\mu' + B_\pm$, in the notation of (3.27). However, in the same notation, the corresponding form of (3.21) for $\Delta v^{(1)}$ reads

$$V(r,z) = -(Ar)^{-1}[H_+ - H_-], \tag{3.46}$$

whence it may be seen immediately that the integral representation of the first term in (3.45), involving $H_\pm$, corresponds exactly to that of the term $\Delta v^{(1)}/r$ in (3.44). The remaining terms in (3.45), involving $J_\pm$, may readily be found by means of the relation (3.31) to match asymptotically with the third term in (3.44), which of course behaves like $\ln r$ for $r \to 0$. Therefore, apart from the integral shown there and other terms which are independent of r, all of which are of a smaller order for $r \to 0$ than those just considered, the expression (3.44) matches with the far-field stress $T^{(1)}_{rr}$, as was to be shown.

To carry the analysis beyond our present discussion of the asymptotic behaviour of the first-order inner fields for large $\bar{r}$, we should obviously require a more complete specification of the shear-stress function t in (2.44). However, such a detailed rheological analysis is beyond the intended scope of the present work, in which we are mainly concerned with the general aspects of the 'zeroth-order' stress field.

Thus, to conclude this section we note that a combined matching of (2.49) and the asymptotic form of $\Delta\bar{u}^{(1)}$ from (2.44) with (3.35) gives, in addition to (3.34) and in lieu of (2.50), the following equation for the particle surface stress $\hat{\tau}^{(0)}$:

$$\zeta = \hat{\tau}^{(0)}\left\{\int_0^{\hat{\tau}^{(0)}} h(\tau)\,d\tau + \frac{1}{\mu-\nu}\ln\alpha + O(1)\right\} \tag{3.47}$$

for $\alpha \to \infty$, where

$$\zeta = \alpha z/\hat{r}. \tag{3.48}$$

In contrast to (2.50), this equation remains valid in the limit of Newtonian shear behaviour, $h = 0$. Moreover, for $h \neq 0$ one concludes that it has the asymptotic solution

$$\hat{\tau}^{(0)} = \hat{\tau}(\zeta)\left\{1 + O\left(\frac{d\hat{\tau}}{d\zeta}\ln\alpha\right)\right\} \tag{3.49}$$

for $\alpha \to \infty$, in which $\hat{\tau}(\zeta)$ denotes the solution to (2.50). The forms of (3.47)–(3.49) suggest that the solution $\hat{\tau}(\zeta)$ will be non-uniformly valid for $z \to 0$, where the assumption of rectilinear shear fails in a region $z = O(\alpha^{-1})$. This is readily seen, for example, in the case of the power-law behaviour (2.51) and (2.52). However, in this case one also encounters fractional powers of z, which leads to singularities at $z = 0$ in the higher-order terms of expansions like those of (3.29)–(3.31), as is shown by the expressions given for these elsewhere, e.g. by Russel & Acrivos (1972).

The effects of such singularities as well as those associated with particle shape, such as bluntness of the particle ends, would doubtless require careful consideration in an analysis of higher-order terms. Barring singularities of particle shape, we expect (3.47) to provide a valid estimate for $\hat{\tau}^{(0)}$ for the shear-dominated rheology postulated in §2.4 above. Then, for shear-thinning fluids, (3.49) should provide a valid asymptotic estimate for the quantity of primary interest here, the particle contribution to stress in a suspension. The latter is given by (3.34) and a further relation, (4.11), to be derived next.

In closing here, we note that at the level of approximation implied by (3.49) one needs only information about the simple-shear behaviour of the fluid. An analysis of higher-order terms would, however, require more complex rheological data, such as the shear modulus $\mu-\nu$ in (3.47), for perturbations to the basic flows.

4. Tensile behaviour of the dilute suspension

The theory of dilute particle suspensions has been thoroughly discussed in the recent literature (Batchelor 1970*a*; Russel & Acrivos 1972), and we shall employ the accepted volume averages for the definition of bulk or macroscopic properties:

$$\langle \mathbf{\Gamma} \rangle = \frac{1}{V}\iiint_V \mathbf{\Gamma}\, dV, \quad \langle \mathbf{T} \rangle = \frac{1}{V}\iiint_V \mathbf{T}\, dV, \tag{4.1), (4.2}$$

which are understood to be taken over both the suspending medium or continuous phase and the particulate phase, with V denoting a representative volume of suspension subject to the macroscopic gradient $\langle \mathbf{\Gamma} \rangle$. In general, V must be large enough to contain a statistically representative collection of particles and also large compared with the typical particle dimensions. Since the analyses referred to above have all dealt with rheologically linear suspending media, it will be necessary in the present work to make certain modifications to the usual methods of calculating volume averages.

To do this, we observe that, given a (sufficiently differentiable) divergence-free tensor field $\mathbf{\Psi}$,

$$\nabla . \mathbf{\Psi} = 0, \tag{4.3}$$

we have, by the divergence theorem, that

$$\iiint_{V_0} \mathbf{\Psi}\, dV = \iint_{A_0} \mathbf{xn}.\mathbf{\Psi}\, dA, \tag{4.4}$$

where A_0 denotes a (sufficiently smooth) closed surface bounding a finite volume V_0 and having unit outward normal $\mathbf{n}(\mathbf{x})$. Since the stress tensor is divergence-free in the type of problem considered here, we can, following Batchelor (1970*a*), make use of (4.4) to put (4.2) into the form

$$\langle \mathbf{T} \rangle = \frac{1}{V}\iint_{\Sigma A_i} \mathbf{xn}.\mathbf{T}\, dA + \frac{1}{V}\iiint_{V-\Sigma V_i} \mathbf{T}\, dV, \tag{4.5}$$

where ΣA_i denotes any set of closed surfaces A_i each lying in the continuous phase and enclosing a volume V_i that contains exactly one particle (the ith), and ΣV_i denotes the set of such volumes.

By choosing the surfaces A_i to coincide with the particle surfaces and recalling that a rheological equation of the form (2.1) applies in the fluid region, one can reduce (4.5) to an expression involving integrals over a typical particle surface plus a volume average of $\mathbf{S}(\mathbf{\Gamma}) - p\mathbf{I}$ over the region occupied by the fluid. In the context of a slender-body analysis the surface integral could be evaluated from the appropriate near-field approximation. However, with the present type of rheology, as opposed to the usual linear cases, the volume average of the nonlinear function $\mathbf{S}$ cannot be directly expressed in terms of the corresponding volume average of its argument $\boldsymbol{\Gamma}$. Thus the integral must be evaluated from a detailed knowledge of the flow field (cf. I). Generally, this might be carried out by integration of an appropriate approximation for the velocity field based, say, on a composite of the near-field and far-field approximations. However, it appears simpler, for the order of slender-body approximation to be considered here, to employ a modification of the technique discussed by Batchelor (1970*a*), which ultimately makes use of the far-field approximation.

Thus we note that the basic stress $\mathbf{T}^{(0)}$ and the first-order approximation in the far field $\mathbf{T}^{(1)}$ are both divergence-free, so that (4.5) applies to these quantities and, as well, to the first-order perturbation

$$\mathbf{T}' \equiv \Delta\mathbf{T}^{(1)} \equiv \mathbf{T}^{(1)} - \mathbf{T}^{(0)}, \tag{4.6}$$

where we employ notation of the form (2.22). Then we may further employ an equation of the form (4.5) for each term on the right-hand side of the identity

$$\langle \mathbf{T} - \mathbf{T}^{(0)} \rangle = \langle \mathbf{T}' \rangle + \langle \mathbf{T} - \mathbf{T}^{(1)} \rangle \tag{4.7}$$

to evaluate $\langle \mathbf{T} - \mathbf{T}^{(0)} \rangle$.

Considering the first term, we recall that $\mathbf{T}'$ is given by the linear rheological equation

$$\mathbf{T}' = \boldsymbol{\eta}^{(0)} : \nabla \mathbf{v}' - p'\mathbf{I}, \tag{4.8}$$

where, as in (4.6),

and

$$\left.\begin{aligned} \mathbf{v}' &\equiv \Delta\mathbf{v}^{(1)} \equiv \mathbf{v}^{(1)} - \mathbf{v}^{(0)} \\ p' &\equiv \Delta p^{(1)} \equiv p^{(1)} - p^{(0)} \end{aligned}\right\} \tag{4.9}$$

denote first-order perturbations and $\boldsymbol{\eta}^{(0)}$ denotes the material tensor (2.24). Finally, and exactly as has been done for the linear isotropic case, we can use the divergence theorem and the relation (4.8) and replace the surface integral in (4.5) by that for a single representative particle to put (4.5) into the form appropriate to a hypothetical anisotropic continuum:

$$\langle \mathbf{T}' \rangle = l^3 \frac{N}{V} \iint_{A_0} [\mathbf{xn}.\mathbf{T}' - \boldsymbol{\eta}^{(0)} : \mathbf{nv}']\, dA - \langle p \rangle' \mathbf{I}, \tag{4.10}$$

where

$$\langle p \rangle' = \frac{1}{V} \iiint_{V - \Sigma V_0} p'\, dV.$$

Here A_0 denotes an arbitrary surface enclosing the sole representative particle, N/V denotes the number of particles per unit volume, while the pressure $\langle p \rangle'$ can be expressed in terms of the volume-average pressure over the fluid.

Now, the surface integral in (4.10), which must be independent of the particular bounding surface A_0, exactly as with other linear continua, is identical with the force doublet or 'stresslet' due to the presence of the particle. Thus, since $\mathbf{T}'$, we recall, was expressly constructed as the stress field due to a line singularity, we may immediately write for the surface integral in (4.10)

$$\langle \mathbf{T}' \rangle + \langle p \rangle' \mathbf{I} = -\mathbf{i}_z \mathbf{i}_z \frac{N}{V} l^3 \int_{-1}^{+1} z f(\alpha; z)\, dz. \tag{4.11}$$

Here f is the axial force distribution of (3.32), and the integral moment represents the magnitude of the effective force doublet (cf. Batchelor 1970*a*).

It remains finally to consider the term $\langle \mathbf{T} - \mathbf{T}^{(1)} \rangle$ in (4.7). As pointed out there, this term can also be expressed in the form (4.5), involving both a surface integral and a volume integral. By choosing the surface A_0 to be far enough removed from the particle surface for the far-field expansion to become valid, we expect the integrand of both the surface and the volume integrals to be approximated by taking

$$\mathbf{T} - \mathbf{T}^{(1)} = \Delta\mathbf{T}^{(2)} \quad \text{for} \quad \alpha \to \infty,$$

where $\Delta\mathbf{T}^{(2)}$ is the second-order perturbation in the far field. Now we observe that the first-order perturbation $\Delta\mathbf{T}^{(1)}$, which represents the effect of a force doublet, behaves like $|\mathbf{x}|^{-3}$ for $|\mathbf{x}| \to \infty$, where $|\mathbf{x}|$ is of course the distance from the particle centre. Provided, then, that $\Delta\mathbf{T}^{(2)} = O(|\mathbf{x}|^{-3})$ for $|\mathbf{x}| \to \infty$, we expect the surface and volume integrals in question to be of a smaller order for $\alpha \to \infty$ than those retained in (4.11), namely $O(\Delta\mathbf{T}^{(2)})$ versus $O(\Delta\mathbf{T}^{(1)})$.

Without carrying out a more detailed investigation of the second-order terms, we shall assume that their large-$|\mathbf{x}|$ behaviour is as required to render (4.11) valid for $\alpha \to \infty$. If so, then we have by (4.7), for the stress quantity of significance, that

$$\langle s_1 \rangle \equiv \langle T_{zz} - T_{\theta\theta} \rangle = s_1^{(0)} + s_1' + o(s_1'),$$

where

$$s_1' = -\frac{N}{V} l^3 \int_{-1}^{+1} f(\alpha; z)\, z\, dz, \tag{4.12}$$

which is identical in form with the corresponding Newtonian result (Batchelor 1971), as is (3.34). Consequently, (4.12) can in general be cast into the more convenient form

$$s_1 - s_1^{(0)} \doteq \phi \langle \sigma \rangle, \tag{4.13}$$

where

$$\phi = \frac{2\pi l^3}{\alpha^2} \frac{N}{V} \int_0^1 \hat{r}^2\, dz \tag{4.14}$$

denotes here the particle volume fraction in the suspension and $\langle \sigma \rangle$ represents an effective volume-average particle stress, given by

$$\langle \sigma \rangle = 2 \frac{\int_0^1 \zeta f^{(0)} \hat{r}^2\, dz}{\int_0^1 \hat{r}^2\, dz} \tag{4.15}$$

with ζ defined by (4.38). We recall that in (4.12) and (4.13), as in related formulae, the stresses have all been rendered dimensionless by division by $\mu^* e_0$.

In the case of the shear-dominated behaviour of §2.4, the expression (3.47) for $f^{(0)}$ applies and, on the basis of the analysis of §3, we assume that it also determines the magnitude of the error in the asymptotic formulae (4.12) and (4.13), for $\alpha \to \infty$. If, further, the fluid exhibits shear-thinning behaviour then the more explicit formula (3.49) applies as well. For example, with the power-law fluid of (2.51) and (2.52), one obtains

$$\langle \sigma \rangle = 2\alpha^{1+1/m} \frac{\int_0^1 (z/\hat{r})^{1+1/m} \hat{r}^2\, dz}{\int_0^1 \hat{r}^2\, dz} [1 + O(\alpha^{1/m-1} \ln \alpha)] \tag{4.16}$$

for $m > 1$.

Compared with the corresponding Newtonian result (Batchelor 1971)

$$\langle\sigma\rangle = \frac{2}{3}\frac{\alpha^2}{\ln\alpha}[1+O(1/\ln\alpha)]\bigg/\int_0^1 \hat{r}^2 dz,$$

(4.16) implies a greatly diminished particle influence for $\alpha\to\infty$, as already suggested by the analysis of I, and as we would expect for more general shear-thinning behaviour.

In conclusion we note that, as usual, there is a restriction on the particle concentrations for which our dilute-suspension formulae are applicable. As already done for the Newtonian case by Batchelor (1971), one can employ far-field expressions like (3.24) to estimate the strength of particle interactions, from which one concludes that (4.12) or (4.13) will be valid for volume fractions ϕ such that $s_1 - s_1^{(0)} \ll 1$; that is, such that the *collective* stress contribution from the particles is small. However, in contrast to the Newtonian case, the present analysis indicates that the stress contribution of an *individual* particle can be intrinsically small, suggesting the same will be true of particle interactions and collective effects in relatively more concentrated suspensions.

This work was partly supported by a National Science Foundation Grant GK 38303. It was presented in part (as paper 8B) at the 46th Annual Society of Rheology Meeting, St Louis, 27 October 1975, and (as MS no. 7737, paper 68d) at the 68th Annual American Institute of Chemical Engineering Meeting, Los Angeles, 19 November 1975. This work was largely done in the University of Michigan Department of Chemical Engineering.

Appendix. Rheological assumptions

Here we record some of the rheological representations and properties that are imputed to the fluid in the preceding analysis. We recall that the Noll fluid, which serves as a general model of an isotropic fluid-like substance, is a material for which the stress tensor for a material particle at the 'present time' can be regarded as a functional on the past history of the velocity gradient for that particle, account being taken of the 'objectivity' or 'frame-indifference' under rigid-body rotations.

We shall not bother here to elaborate on this notion mathematically, since several existing treatises (Truesdell 1966; Coleman, Markovitz & Noll 1966; Pipkin 1972) cover the subject thoroughly. Rather, we merely adopt outlines of the methods employed by others, e.g. by Coleman *et al.* (1966), to deduce the form of stress tensor used in the present work. Specifically, we observe that the axisymmetric velocity gradient $\mathbf{\Gamma}$ given by (2.28) can be expressed alternatively as

$$\mathbf{\Gamma} = \gamma_1 \mathbf{N} + \gamma_2 \mathbf{N}^{\mathrm{T}} + e_1 \mathbf{N}.\mathbf{N}^{\mathrm{T}} + e_2 \mathbf{N}^{\mathrm{T}}.\mathbf{N} - e_3 \mathbf{I}, \tag{A 1}$$

where the subscripts 1, 2 and 3 refer respectively to z, r and θ, as used in the text above, and where $\mathbf{N}$ and its transpose $\mathbf{N}^{\mathrm{T}}$ are the nilpotent 'dyads'

$$\left.\begin{array}{c} \mathbf{N} = \mathbf{i}_1\mathbf{i}_2, \quad \mathbf{N}^{\mathrm{T}} = \mathbf{i}_2\mathbf{i}_1, \\ \mathbf{i}_2.\mathbf{i}_1 = 0, \quad \mathbf{N}^2 \equiv \mathbf{N}.\mathbf{N} = \mathbf{0}, \quad (\mathbf{N}^{\mathrm{T}})^2 \equiv \mathbf{N}^{\mathrm{T}}.\mathbf{N}^{\mathrm{T}} = 0. \end{array}\right\} \tag{A 2}$$

with

These are familiar in the theory of viscometric flows for which the e's all vanish and one of the shear rates may, without loss of generality, be taken as identically zero.

Thus, with the past history of the velocity gradient being represented by (A 1) for every material particle, we can state that the stress tensor $\mathbf{T}$ at any particle is a (symmetric) function of the tensor $\mathbf{N}$ and a functional on the scalar coefficients in (A 1), which we denote by (γ, e). Therefore, because of the nil-potency of $\mathbf{N}$ it follows, exactly as with viscometric flows, that the most general form for $\mathbf{T}$ is

$$\mathbf{T} = \mathscr{t}(\mathbf{N}+\mathbf{N}^{\mathrm{T}}) + \mathscr{s}_1 \mathbf{N}.\mathbf{N}^{\mathrm{T}} + \mathscr{s}_2 \mathbf{N}^{\mathrm{T}}.\mathbf{N} - \mathbf{I}, \tag{A 3}$$

where, for the incompressible fluid assumed here, p is a rheologically indeterminate (or 'dynamic') pressure, and $\mathscr{t}$, $\mathscr{s}_1$ and $\mathscr{s}_2$ are scalar functionals on the history of the scalars (γ, e).

Quasi-steady flow

In the case of a materially steady flow the scalars (γ, e) are independent of time when viewed from the history of any given material particle and, up to a time-dependent rotation or 'change of frame', the form (A 1) represents in effect a so-called flow of 'constant stretch history' (Truesdell 1966; Pipkin 1972). In this case, we may regard the functionals $\mathscr{t}$, $\mathscr{s}_1$ and $\mathscr{s}_2$ as functions t, s_1 and s_2 of the 'present' or local, instantaneous values of (γ, e) at a given particle, as was done in the quasi-steady approximation of § 2.

As an approximation, the assumption of quasi-steady flow can in principle be based on the smallness of the material time rates of change of $\mathbf{\Gamma}$ (or, more precisely, the rates of change of its symmetric part, the rate-of-deformation tensor $\mathbf{E}$, with allowance being made for changes of frame), provided that one has recourse to a theory of stress relaxation or 'fading memory' (Coleman & Noll 1961).

For the purposes of the present work, we assume simply that this can be made to correspond to a criterion of the form

$$\lambda\,|\dot{e}| \ll |e|, \tag{A 4}$$

where λ denotes a relaxation time for the fluid, while e denotes any of the quantities e and γ in (A 1) and $\dot{e}$ refers to a 'substantial' or material rate of change. Given the basic far-field flow (1.3), the criterion (A 4) represents, first of all, a restriction on the time rate of change of the far-field extension rate e_0. In addition, and to ensure that convective rates of change are small throughout the flow field, one concludes by means of elementary dimensional considerations that (A 4) will lead to a further restriction, of the global form

$$\lambda e_0 \ll 1. \tag{A 5}$$

Now, if interpreted in too strict a sense (A 5) would imply a restriction to the Newtonian régime. However, a little reflexion leads one to conclude that (A 5) should in fact be interpreted in a local sense, with λ denoting an effective relaxation time for the local kinematics. As pointed out in an earlier analysis (I),†

† However, the kinematical justifications given in that analysis for the assumption of quasi-steady viscometric behaviour are not correct.

there is evidence to suggest that in real fluids the effective relaxation times for flows with large deformation rates may be much smaller than those associated with a rest state, that is, with 'linear viscoelasticity'. Interpreted in this light, (A 5) may in reality represent a much milder restriction on e_0, which in the present context allows quasi-steady and simultaneous nonlinear behaviour.

Rheological functions

The functional forms of the coefficients t, s_1 and s_2 in (A 3) and their quasi-steady counterparts t, s_1 and s_2 are restricted by the principle of objectivity or frame indifference. For the quasi-steady case one also has recourse to the theory of flows with constant stretch history, and the associated representations of an isotropic tensor function.† However, for the relatively elementary form (A 1) one can establish, in a rather direct way, the isotropic representation

$$\mathbf{T} = 2\{\eta\mathbf{E} - \zeta\mathbf{E}' + \xi\mathbf{E}^2\} - p\mathbf{I}, \tag{A 6}$$

where, with $\mathbf{E}$ and $\mathbf{\Omega}$ defined as in (3.2), $\mathbf{E}^2 = \mathbf{E}.\mathbf{E}$ and

$$\mathbf{E}' = \dot{\mathbf{E}} + \mathbf{E}.\mathbf{\Omega} - \mathbf{\Omega}.\mathbf{E}. \tag{A 7}$$

The scalar coefficients η, ζ and ξ in (A 6) are functions of the joint (isotropic) scalar invariants of the kinematic tensor $\mathbf{E}$ and $\mathbf{E}'$, and it will be noted that the tensor $\mathbf{E}'$ is nothing more than the well-known Jaumann derivative of $\mathbf{E}$. In the frame of reference associated with the representation (A 1) the material derivative $\dot{\mathbf{E}}$ is understood to vanish, which would not be the case for other frames, differing by an arbitrary unsteady rotation.

The representation (A 6) can be derived by noting that the three kinematic tensors involved can be expressed linearly in terms of the unit tensor $\mathbf{I}$ and the three symmetric tensors $\mathbf{N}+\mathbf{N}^{\mathrm{T}}$, $\mathbf{N}.\mathbf{N}^{\mathrm{T}}$ and $\mathbf{N}^{\mathrm{T}}.\mathbf{N}$ of (A 1), and hence that the latter can be linearly related to the former (all being regarded, if one chooses, as elements in a linear space of symmetric tensors).

It is interesting to note that, as with (A 3), the form (A 6) is similar to the representation for stress associated with viscometric flows (cf. Goddard 1967) and, as such, could equally well have been expressed in terms of the first two Rivlin–Ericksen tensors.

We further observe that the quasi-steady forms t, s_1 and s_2 of the rheological functions in (A 3) can be expressed as

$$\left.\begin{aligned} t &= \eta(\gamma_1+\gamma_2) - \zeta(\gamma_1-\gamma_2)(e_1-e_2) + \xi(e_1+e_2)(\gamma_1+\gamma_2), \\ s_1 &= 2\eta(e_1-e_3) + \zeta(\gamma_1-\gamma_2)(\gamma_1+\gamma_2) + 2\xi[e_1^2 - e_3^2 + \tfrac{1}{4}(\gamma_1+\gamma_2)^2], \\ s_2 &= 2\eta(e_2-e_3) - \zeta(\gamma_1-\gamma_2)(\gamma_1+\gamma_2) + 2\xi[e_2^2 - e_3^2 + \tfrac{1}{4}(\gamma_1+\gamma_2)^2], \end{aligned}\right\} \tag{A 8}$$

† In which, we recall, 'Wang's theorem' (Truesdell 1966) tells us that the stress can be represented as an isotropic tensor function of the first three Rivlin–Ericksen tensors, also implying an alternative function of the rate of deformation and its first two Jaumann derivatives.

where, for incompressible fluids, $e_3 = -(e_1+e_2)$. In this case, the rheological coefficients η, ζ and ξ depend on the set of invariants

$$\begin{aligned}\mathscr{I} &= \operatorname{tr}\{\mathbf{E}^2, \mathbf{E}^3, \mathbf{E}'.\mathbf{E}, \mathbf{E}'^2, \ldots\} \\ &= 2\{[e_1^2+e_2^2+e_1e_2+\gamma^2], [\tfrac{3}{2}(e_1+e_2)(\gamma^2-e_1e_2)], [\iota(\gamma_2-\gamma_1)(e_1-e_2)\gamma], \\ &\quad (\gamma_2-\gamma_1)^2[\gamma^2+\tfrac{1}{4}(e_1-e_2)^2], \ldots\},\end{aligned} \tag{A 9}$$

where $\gamma \equiv \frac{1}{2}(\gamma_1+\gamma_2)$ and where tr{ } denotes the set of traces of the tensors in the set { }. Without attempting to establish the complete minimal set of independent invariants, we simply note several properties of the terms shown explicitly in (A 8) and (A 9) that are pertinent to the limiting rheological behaviour of the fluid. First of all, for viscometric flows, where $e_1 = e_2 = 0$ and, say, $\gamma_2 = 0$, such that the sole independent invariant is proportional to γ_1^2, we recover the well-known result that η, ζ and ξ must be even functions of γ_1.

Second, and as regards the assumption of shear-dominated behaviour in §2.4, we note that if $\alpha \to \infty$, with

then
$$\left.\begin{aligned}\gamma_1 = O(\alpha), \quad \gamma_2 = O(1/\alpha), \quad e_1 = O(1), \quad e_2 = O(1),\\ \mathscr{I} \sim 2\{\gamma_1^2, \tfrac{3}{8}\gamma_1^2(e_1+e_2), \tfrac{1}{2}\gamma_1^3(e_2-e_1), \tfrac{1}{4}\gamma_1^4, \ldots\}.\end{aligned}\right\} \tag{A 10}$$

Thus, in addition to the terms in e_1 and e_2 shown explicitly in (A 8), the invariants $\mathscr{I}$ and, hence, the functions η, ζ and ξ will depend asymptotically on e_1 and e_2 in the above limit. Therefore, the earlier postulate (I) of shear-dominated behaviour implies a very special class of fluid models.

Finally, we consider the derivatives of (A 8), which govern stress perturbations in the basic far-field flow, in order to establish the validity of (2.32) and the relation between the representations (A 8) and (3.1). For the most part, it is a straightforward matter to evaluate the various first-order derivatives of the rheological functions in (A 8) for the basic flow $e_1 = e_0$, $e_2 = e_3 = -\frac{1}{2}e_0$, $\gamma_1 = \gamma_2 = 0$. Thus, for example, one has directly from (A 8)

$$\left(\frac{\partial s_1}{\partial \gamma_1}\right)_0 = 3e_0\left[\left(\frac{\partial \eta}{\partial \gamma_1}\right)_0 + \frac{e_0}{2}\left(\frac{\partial \xi}{\partial \gamma_1}\right)_0\right], \tag{A 11}$$

where the subscript zero denotes, of course, the value taken in the basic flow under consideration. However, the functions η and ξ depend on the invariants (A 9), which are homogeneous polynomial forms, of degree two at least, in (γ, e). It follows by the chain rule that the γ_1 derivatives in (A 11) must vanish in the basic flow, provided that η and ξ are analytic in the invariants.

Hence, with this assumption of smoothness near the far-field state, one can confirm the remaining relations for the derivatives in (2.32). In addition, one finds that the material constants in (3.1) are given in terms of the functions (A 8) by

$$\left.\begin{aligned}\mu = \eta_0 + \tfrac{1}{2}\xi_0 e_0, \quad \nu = \tfrac{3}{2}\zeta_0 e_0,\\ \mu' = \eta_0 + \frac{e_0}{12}\left[\frac{\partial \eta}{\partial e_1} + \frac{\partial \eta}{\partial e_2}\right]_0, \quad \mu'' = \eta_0 - \xi_0 e_0.\end{aligned}\right\} \tag{A 12}$$

We recall that, with the convention of (2.3), these would be scaled by a factor of μ^*. It is interesting to observe that the shear stress is independent of vorticity in (3.1) only if $\zeta_0 = 0$, which, as mentioned in §3.1, will be the case for certain restricted fluid models, such as the Reiner–Rivlin fluid, for which (A 6) with $\zeta \equiv 0$ applies to *any* flow.

REFERENCES

BATCHELOR, G. K. 1970*a* The stress system in suspensions of force-free particles. *J. Fluid Mech.* **41**, 545.

BATCHELOR, G. K. 1970*b* Slender-body theory for particles of arbitrary cross-section in Stokes flow. *J. Fluid Mech.* **44**, 419.

BATCHELOR, G. K. 1971 The stress generated in a non-dilute suspension of elongated particles by pure straining motion. *J. Fluid Mech.* **46**, 813.

BURGERS, J. M. 1938 In *Second Report on Viscosity and Plasticity. Kon. Ned. Akad. Wet. Verhand.* (*Eerste Sectie*), **16**, 113.

COLEMAN, B. D., MARKOWITZ, H. & NOLL, W. 1966 *Viscometric Flows of non-Newtonian Fluids*. Springer.

COLEMAN, B. D. & NOLL, W. 1961 Foundations of linear visco-elasticity. *Rev. Mod. Phys.* **33**, 239.

COX, R. G. 1970 The motion of long slender bodies in a viscous fluid. Part 1. General theory. *J. Fluid Mech.* **44**, 791.

COX, R. G. 1971 The motion of long slender bodies in a viscous fluid. Part 2. Shear flow. *J. Fluid Mech.* **45**, 625.

DODSON, A. G., TOWNSEND, P. & WALTERS, K. 1974 Non-Newtonian flow in pipes of non-circular section. *Computers & Fluids*, **2**, 317.

ERICKSEN, J. L. 1958 Overdetermination of speed in rectilinear motion of non-Newtonian fluids. *Quart. Appl. Math.* **14**, 318.

FRAENKEL, L. E. 1969 On the method of matched asymptotic expansions. II. *Proc. Camb. Phil. Soc.* **65**, 233.

FREDRICKSON, A. G. 1964 *Principles and Applications of Rheology*. Prentice-Hall.

GREEN, A. E. & RIVLIN, R. S. 1956 Steady flow of non-Newtonian fluids through tubes. *Quart. Appl. Math.* **14**, 299.

GREEN, A. E. & ZERNA, W. 1954 *Theoretical Elasticity*. Oxford: Clarendon Press.

GODDARD, J. D. 1967 A modified functional analysis for visco-elastic fluids. *Trans. Soc. Rheol.* **11**, 381.

GODDARD, J. D. 1975 Tensile stress contributions of oriented rod-like particles in non-Newtonian fluids. *J. Non-Newtonian Fluid Mech.* (to appear).

GODDARD, J. D. & MILLER, C. 1967 Non-linear effects in the rheology of dilute suspensions. *J. Fluid Mech.* **28**, 657.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

INCE, E. L. 1956 *Ordinary Differential Equations*. Dover.

KARDOS, J. L. 1973 Structure property relationships in short-fiber reinforced plastics. *Crit. Rev. Solid State Sci.* **3**, 419.

LAMB, H. 1945 *Hydrodynamics*, 6th ed. Dover.

LEAL, L. G. 1975 The slow motion of slender road-like particles in a second-order fluid. *J. Fluid Mech.* **69**, 305.

LOVE, A. E. H. 1944 *A Treatise on the Mathematical Theory of Elasticity*, 4th ed. Dover.

MEWIS, J. & METZNER, A. B. 1974 The rheological properties of suspensions of fibres in Newtonian fluids subjected to extensional deformations. *J. Fluid Mech.* **62**, 593.

PIPKIN, A. C. 1972 *Lectures on Viscoelasticity Theory*. Springer.

RUSSEL, W. R. & ACRIVOS, A. 1972 On the effective moduli of composite materials. *J. Appl. Math. Phys.* **23**, 434. (See also 1973, **24**, 582–600, 838–852.)

Synge, J. L. 1957 *The Hypercircle in Mathematical Physics*. Cambridge University Press.

Tillet, J. P. K. 1970 Axial and transverse stokes flow past slender axisymmetric bodies. *J. Fluid Mech.* **44**, 401.

Truesdell, C. 1966 *The Elements of Continuum Mechanics*. Springer.

Weinberger, C. B. & Goddard, J. D. 1974 Extensional flow behaviour of polymer solutions and particle suspensions in a spinning motion. *Int. J. Multiphase Flow*, **1**, 465.

Willis, J. R. 1965 The elastic interaction energy of dislocation loops in anisotropic media. *Quart. J. Mech. Appl. Math.* **18**, 419.